Searching for Chaos

Math Circles – Washington University

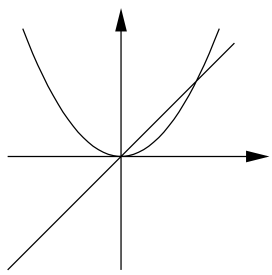
Geir Arne Hjelle April 22, 2007

Today we will study the function $f(x) = x^2 + c$ for different values of c. The expression x^2 is pronounced "x squared", and simply means that x is multiplied by itself. We could also have written the function as $f(x) = x \cdot x + c$. The graph of our function takes the shape of something called a parabola.

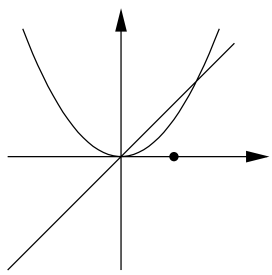
1. First we will study the function $f(x) = x^2$ (with $c = 0$).

   - Draw a parabola in the coordinate system with the opening upwards, symmetric about the y-axis, and its bottom on the x-axis.

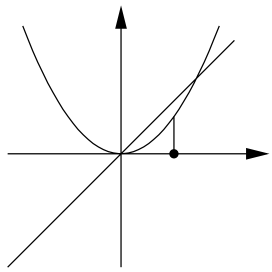
     (This position of the bottom of the parabola corresponds to the value of c, so here we are setting $c = 0$.)



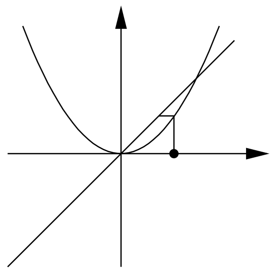
   - Choose a point along the x-axis, and mark it with a small circle.



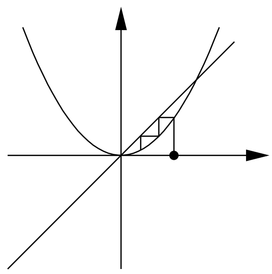
- Draw a vertical line from your chosen point to the parabola.



- Draw a horizontal line from the parabola to the diagonal line.



- Repeat the procedure several times; draw a vertical line from the point on the diagonal to the parabola, and then a horizontal line to the diagonal.



- What happens to your point as you repeat the procedure? If you choose a different starting point, will the same thing happen?

Later we will refer to the process explained above as "drawing a web-diagram for the function".

2. Draw a web-diagram for $f(x) = x^2 + 0.5$.

   - To do this, we need to draw a parabola in the coordinate system with its bottom at the line 0.5 units above the x-axis. In the given coordinate systems, that means 5 small squares above the x-axis.
   - What will happen for other $c > 0.5$? What is the limiting c for this behavior?

3. Draw a web-diagram for $f(x) = x^2 - 0.5$.
   - What happens if you choose a starting point directly above/below (or at) one of the points where the parabola and the diagonal line intersect?
   - Why is the behavior at the two fixed points different?
4. Draw a web-diagram for $f(x) = x^2 - 1$.
   - What happens? Why is our point attracted to 0 and 1?
5. Draw a web-diagram for $f(x) = x^2 - 1.5$.
   - This should look chaotic! Can we still say something. Are there any points that we surely will not visit?
6. Draw a web-diagram for $f(x) = x^2 - 2$.
   - This is true chaos!
7. Draw a web-diagram for $f(x) = x^2 - 1.75$.
   - This one is hard, you need to be precise to see the correct behavior (that's because we are dealing with chaos).
   - We have found a window of periodicty within the chaos!
8. Can you think of a way to summarize all our findings in one diagram?

The programs we have used today are available on the internet at

`http://www.math.wustl.edu/~hjelle/software/`.