

Modular Arithmetic and Doomsday

1. Divisibility tests. Can you see why these work?
 - (a) A number is divisible by two if and only if the last digit is even.
 - (b) A number is divisible by five if and only if the last digit is a 0 or 5.
 - (c) A number is divisible by nine if and only if the sum of the digits is divisible by 9.
 - (d) A number is divisible by four if and only if the last two digits (taken as a two digit number) are divisible by 4.
 - (e) Divisibility by 11?
 - (f) Divisibility by 7?
 - (g) Divisibility by 13?
2. *The Census-Taker*. A census-taker knocks on a door and asks the woman inside how many children she has and how old they are.

“I have three sons, their ages are whole numbers and the product of their ages is 36,” says the mother.

“That’s not enough information,” responds the census-taker.

“I’d tell you the sum of their ages, but you’d still be stumped.”

“I wish you would tell me something more.”

“Okay, my oldest son, Luke, likes dogs.”

What are the ages of the sons?

3. Look at the two-digit multiples of 9 (9,18,27, etc.). Note that the sum of these digits are all 9. Why is this true. Does this generalize to other multiples of 9?
4. *Subtraction Magic Trick*. While blindfolded, a magician asks a member from the audience to write a 10 digit number on the board. The magician then asks the volunteer to reverse the digits and subtract the smaller number from the larger number. Then, the volunteer circles any one of the digits of the result (except, the volunteer is not allowed to circle 0) and read all the digits to the magician. The magician then tells the audience what the circled digit is. How is this trick done? Why does the trick work? By the way, there is nothing special about the number of digits, the number could have been any length.

5. *John Conway's Doomsday Algorithm.* This is a method to determine the day of the week quickly without a calendar or calculator. I will tell you how the algorithm works for dates in the 1900s. Your job is to learn the algorithm and figure out why it works.

- For each year, doomsday is defined to be the day of the week that the last day of February falls on (2/28 for ordinary years, 2/29 for leap years).
- For any year, the dates 4/4, 6/6, 8/8, 10/10 and 12/12 are doomsdays.
- For any year, the dates 5/9, 9/5, 7/11, 11/7 are doomsdays (“A 9 to 5 job at the 7-11”)
- The “last” day of January is a doomsday, if we define this to be 2/1 in a leap year.
- Facts to remember: doomsday for 1900 is Wednesday, doomsday for 2000 is Tuesday.
- All days are regarded as modulo 7, with Sunday equal to 0 (you can remember this as noneday, with nuns in a church). Therefore Monday is 1, Tuesday is 2, etc.

Here is how you compute the day of the week for a given date (with the example being September 30, 2007).

- I. Determine the number of days (in $(\text{mod } 7)$) that the date is from a doomsday.
 - For 9/30/07 we are -10 from the doomsday of 10/10, which is equal to 4 $(\text{mod } 7)$.
 - II. Add in for the century: $+3$ (Wednesday) for 1900, $+2$ (Tuesday) for 2000.
 - $+2$ since we are in the year 2007.
 - III. Dozen: divide the year of concern by 12, write down the quotient.
 - Our year is 07, $7/12$ is 0 with remainder 7.
 - IV. Remainder of division by 12
 - For the year 07, the remainder was 7.
 - V. Divide the remainder in Part IV by 4, write down the quotient.
 - $7/4$ is 1 with a remainder of 3.
 - VI. Add everything up $(\text{mod } 7)$
 - $4 + 2 + 0 + 7 + 1 = 14 \equiv 0 \pmod{7}$, so September 30, 2007 is a Sunday!
- (a) Practice this with the following dates:
- July 17, 1970 (Friday)
 - Christmas day in 2047 (Wednesday)
 - Your birthday this year
 - Your birthday
- (b) Why does it work???
- (c) What about for other centuries?

For more information see http://en.wikipedia.org/wiki/Doomsday_algorithm