# Problem Solving Methods

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One of the main points of problem solving is to learn techniques by just doing problems. So, lets start with a few problems and learn a few techniques.

## Patience

1. Find a 9-digit number using each digit 1 through 9 once, such that the first $n$ digits are divisible by $n$
2. Sheep and Wolves. On a $5 \times 5$ chessboard place 5 wolves (who move like chess queens) and 3 sheep so that the sheep are safe from being eaten by the wolves.
3. Triangle problem: arrange the numbers 1 through 6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it. For example

   6 4 1<br>
   2 3<br>
   1

   almost works but it has two 1's and no 5.

   How about with 10 numbers? 15?

## Try special cases (make up an easier problem!)

4. How many zeros are at the end of the number

$$100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$$

5. The numbers 1 through 100 are written on the board. Take two numbers, $u$ and $v$ and erase them writing $uv + u + v$ in their place. After a while, there will only be one number left on the board. What are the possible numbers left?

# Getting dirty

6. What is the smallest number that can not be written by subtracting a prime from a square. For example

$$1 = 4 - 3$$
$$2 = 9 - 7$$
$$3 = ?$$

(How about the next smallest number?)

7. In the year 1971, Smith said, "I was $n$ years old in the year $n^2$." When was Smith born?

8. For every positive integer $n$, look at the number $n^3 - n$. The first few are here:

| $n$ | $n^3 - n$ |
|---|---|
| 1 | 0 |
| 2 | 6 |

Keep filling out this chart. For at least the first few numbers in the $n^3 - n$ column, they should be divisible by 3.

(a) Are all the numbers $n^3 - n$ divisible by 3?

(b) If not, find one that is not. If so, show that this is always the case.

9. For every positive integer $n$, look at the number $n^5 - n$. The first few are here:

| $n$ | $n^5 - n$ |
|---|---|
| 1 | 0 |
| 2 | 40 |

Keep filling out this chart. For at least the first few numbers in the $n^5 - n$ column, they should be divisible by 5.

(a) Are all the numbers $n^5 - n$ divisible by 5?

(b) If not, find one that is not. If so, show that this is always the case.

# Organization

10. What is the greatest number of regions into which three straight lines (of infinite extent) can divide the plane? How about 4 lines? How about $n$ lines?

11. What do you get if you add up all the numbers from 1 to 100? Can you do this by using any tricks? What if you just add them up directly?

12. Fill out the multiplication table below.

| $\times$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | | | | | |
| 5 | | | | | | | | | | |
| 6 | | | | | | | | | | |
| 7 | | | | | | | | | | |
| 8 | | | | | | | | | | |
| 9 | | | | | | | | | | |
| 10 | | | | | | | | | | |

What do you get if you add up all the numbers in the table?

13. How many rectangles are in a $10 \times 10$ rectangle?

14. Magic Squares: Imagine a $3 \times 3$ array of squares:

| | | |
|---|---|---|
| | | |
| | | |
| | | |

You are to put numbers, 1 through 9, one in each square, so that each row and column add up to the same number.

(a) What are the possibilities for the sum of the rows and columns?

(b) What are the possible ways to add three numbers between 1 and 9 to the sum you found in the previous part?

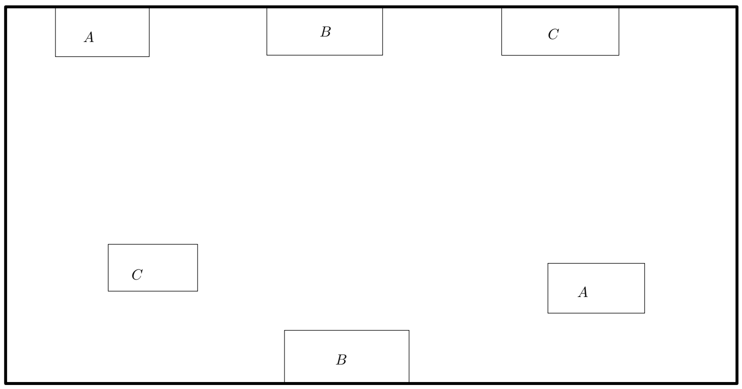
(c) Find a $3 \times 3$ magic square.

(d) Can you turn your $3 \times 3$ magic square into a better magic square by making sure the diagonals also have the same sum?

(e) What about $2 \times 2$ magic square? $4 \times 4$?

# Wishful thinking

15. Connect $A$ to $A$, $B$ to $B$ and $C$ to $C$ without crossing lines or leaving the box.




# Symmetry

Many of the problems here can use symmetry. For example, in the triangle problem (Problem 3), it is helpful to realize that the following triangles are essentially the same:

| | |
|---|---|
| 6 4 1 | 1 4 6 |
| 2 3 | 3 2 |
| 1 | 1 |

Another example is the problem of the sheep and the wolves (Problem 2). In this problem, there is a symmetry between the wolves and the sheep. The case with 5 wolves and 3 sheep is equivalent to the problem with 3 wolves and 5 sheep (do you see why?).

16. In the game of tic-tac-toe, how many different first moves are there?

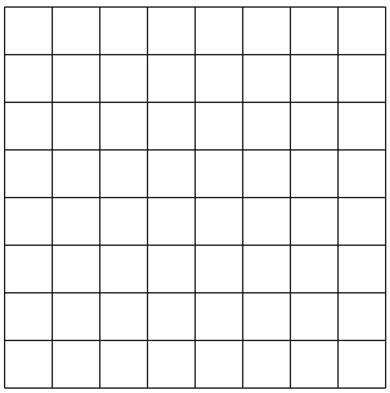
    For each of the first moves, how many second moves are there?

# Invariants

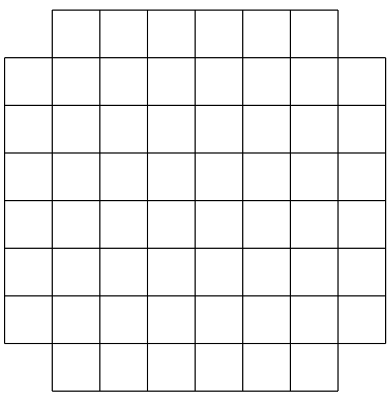
17. Here are a few classic coloring problems.

    (a) Take an $8 \times 8$ board (containing 64 squares) and cover it with 32 dominoes of size $2 \times 1$, so each domino covers 2 adjacent squares.

    (Side note: In 1961 the British physicist M.E. Fisher showed that there are $12,988,816$ ways to do this.)

    (b) Now take two opposite corners off the board and cover it with 31 dominoes:

18. Let $n$ be an odd number and write the numbers $1, 2, \ldots 2n$ on the board. Then, pick any two numbers $a$,$b$ and erase them and replace them with $|a - b|$. Do you end up with an odd number or an even number or does it depend?

# Other Problems

19. Imagine a wooden cube with side length of 3 inches. Imagine cutting the cube into 27 smaller cubes of side length 1, using straight cuts (think of a Rubik's cube). What is the minimum number of straight cuts needed to make this happen?

    What if it is a $4 \times 4 \times 4$ cube? What is the minimum number of cuts? $5 \times 5 \times 5$?