Polyhedra and Platonic Solids

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We are going to build our polyhedra up one dimension at a time.

If we start with 0 dimensions, we have vertices. These are just points (think of a dot on a piece of paper). We can also call a vertex a 0-cells. All vertices are congruent, which means if you moved one vertex on top of another they would look the same.

In dimension 1, we have edges, or line segments. We can call an edge a 1-cell. At the end of each edge is a vertex.

1. Is every edge congruent to every other edge? Why or why not?

Polygons

In dimension 2 we have **faces** which are built out of edges connected at their vertices. We can call a face a 2-cell.

2. Draw some edges (line segments) on your paper and connect them at their vertices. Edges are not allowed to cross. What are some possibilities?

3. Draw some edges on your paper. This time each edge must be connected to exactly on other edge, at the vertex. Again, edges are not allowed to cross. What are the possibilities?

4. What you drew in Problem 3 is called a **polygon**. Draw some polygons with these properties:

   (a) A polygon is **convex** if every line connecting two points of the polygon lies within the polygon. Draw a convex polygon and a non-convex polygon.

   (b) A polygon is **cyclic** if every vertex lies on a single circle. Draw a cyclic polygon.

   (c) A polygon is **equiangular** if the angle at each vertex is the same for all vertices. Draw an equiangular polygon.

(d) A polygon is **equilateral** if all edges are the same length. Draw an equilateral polygon.

(e) A polygon is **regular** it is both cyclic and equilateral. Draw an regular polygon polygon. Draw an equilateral polygon that is not regular. Draw an equiangular polygon that is not regular.

(f) For each of the polygons you drew, determine which of the above properties they satisfy.

5. **Naming polygons:** Polygons are named by Greek prefixes. Here are the Greek prefixes:

1	2	3	4	5	6	7	8	9	10	12	20
mono	di	tri	tetra	penta	hexa	hepta	octa	ennea	deca	dodeca	icosa

To name polygon, take the Greek prefix for the number of sides and add "gon" to the end. For example, the polygon with only one side would be a monogon (what would this look like?). Fill in the chart:

n	**Name of n-gon**
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
12	
20	

6. Consider a regular triangle (also called an equilateral triangle). What are the angles at each vertex? Why?

(a) What are the angles at each vertex of a regular quadrilateral (square)?

(b) What are the angles at each vertex of a regular pentagon?

(c) What are the angles at each vertex of a regular hexagon?

(d) What are the angles at each vertex of a regular heptagon?

(e) What are the angles at each vertex of a regular octagon?

(f) What are the angles at each vertex of a regular enneagon?

(g) What are the angles at each vertex of a regular n-gon?

Polyhedra

7. We will use polygons as **faces** to build up **polyhedra**. Basically, you want to connect polygons together at their edges so that each edge is contained in exactly two polygons. The result will be a polyhedra. Draw a polyhedra.

8. Take two polygons which are congruent but raise one above the other. Then connect the vertices with vertical edges. The result is a **prism**. Draw a few prisms.

9. As in Problem 8, take two congruent polygons and raise one above the other, but this time with the upper vertices above the middle of the lower edges. Now connect the upper vertices with the two vertices in the edge below. This is an **anti-prism**. Draw a few anti-prisms.

10. Take any polygon and put a vertex above the polygon. Connect each of the vertices in the polygon with the new vertex. This gives a pyramid. Draw a few different pyramids.

11. As with polygons, a polyhedron is **convex** if every line connecting two points of the polyhedron is completely contained inside the polyhedron. Draw a non-convex polyhedron.

12. **Naming polyhedra:** As with polygons, polyhedra are named by Greek prefixes. Thus, to name polyhedron, take the Greek prefix for the number of faces and add "hedron" to the end. For example, the polyhedron with only one side would be called a monohedron (what would this look like?). Fill in the chart:

n	Name of n-sided polyhedron
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
12	
20	

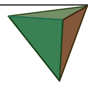
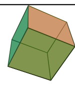
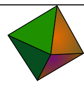
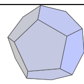
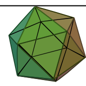
13. **Platonic Solids**: A polyhedron is Platonic if every face is a congruent regular polygon and the same number of faces meet at every vertex.

    An example of a Platonic solid is a cube. Draw a cube and explain why it is platonic.

14. Fill in the table below. It will be helpful to have a model of each solid. There are great cut-out paper models you can find at one of these web-pages:

    ```
    http://www.mathsisfun.com/platonic_solids.html
    http://www.korthalsaltes.com/pdf/all_paper_models_v2.PDF
    ```

    - Count the number of vertices.
    - Count the number of edges.
    - Count the number of faces.
    - The Schläfli symbol is the number of sides on each face and the number of faces for each vertex. For example, the cube has faces with 4 sides and vertices with 3 faces, so the Schläfli symbol is $\{4, 3\}$.
    - The vertex configuration is the sequence of numbers representing the number of sides of the faces going around the vertex. So, for a cube, the vertex configuration is $4, 4, 4$ because there are 3 faces each with 4 sides.

Polyhedron	**Vertices**	**Edges**	**Faces**	**Schläfli symbol**	**Vertex configuration**
Tetrahedron 					
Hexahedron 					
Octahedron 					
Dodecahedron 					
Icosahedron 					

15. Add a couple lines to your table for some of your other polyhedra. Fill in all the data for that line.

16. In a polyhedron, what is the smallest number of faces meeting at a vertex? Why?

17. How many regular triangular faces can meet at a vertex in a convex polyhedron? Why?

18. How many regular quadrilateral (square) faces can meet at a vertex in a convex polyhedron? Why?

19. How many regular pentagonal faces can meet at a vertex in a convex polyhedron? Why?

20. How many regular hexagonal faces can meet at a vertex in a convex polyhedron? Why?

21. What constraints are there for the faces of a Platonic solid meeting at a vertex?

22. What are the possible vertex configurations for any possible Platonic solid?

Faces	**Possible Num of faces at a vertex**	**Possible Vertex configurations**
Triangles		
Squares		
Pentagons		
Hexagons		

23. What are the platonic solids for each possibility you found in Problem 22?

24. The **dual** of a convex polyhedron is obtained by placing a vertex in the center of each face. Then, the vertices are connected with an edge if there is an edge between the corresponding faces.

    (a) What are the duals of the five Platonic solids?

    (b) What are the duals of the duals of the five Platonic solids?

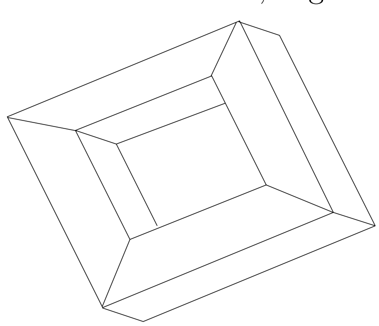
Euler Characteristic

25. Take some of the polyhedra you have used and add them to the chart below. Fill out the chart.

Polyhedron	**Vertices**	**Edges**	**Faces**	
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

26. Find a relationship between the numbers in the chart in Problem 25.

27. Consider a "polyhedra" that is not spherical but rather toroidal. (In other words it has a hole in it, like a torus, an example is below.) What sort of relationship can you find for the number of vertices, edges and faces, for these polyhedra?



Angular Deficit

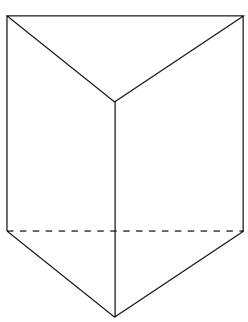
28. Take a vertex and add up the angles of the faces. If the vertex is **planar** then the sum of these faces will be 360° Think about 6 equilateral triangles arranged around a vertex.

    In a convex polyhedra, the angles at each vertex will not add up to 360°, but will add to something less. The angular deficit at a vertex is $360 - (\text{sum of angles})$.

    In your Platonic solids, since each vertex is the same, the angular deficit is the same of each vertex. What is it?

Polyhedron	**Angular Deficit at Each Vertex**
Tetrahedron	
Cube	
Octahedron	
Dodecahedron	
Icosahedron	

29. Take the polyhedra below where every face is either an equilateral triangle or a square (the polyhedra is a triangular prism), see below. What is the angular deficit at each vertex?



30. For any polygon, the total angular deficit is the deficit sum of the deficit at each vertex. For example, in the triangular prism, every vertex has deficit of 120° and there are 6 vertices. Thus the total angular deficit is $6 \cdot 120 = 720$.

    Find the total angular deficit for your Platonic solids.

Polyhedron	Total Angular Deficit
Tetrahedron	
Cube	
Octahedron	
Dodecahedron	
Icosahedron	

31. Find the total angular deficit for more polyhedra. See if you notice any patterns?

32. How about total angular deficit for non-convex polyhedra like the **small stellated dodecahedron**. This is a dodecahedron with a pyramid on each pentagonal face (you can consider the triangles to all be equilateral if you like). What is the total angular deficit?

    What if instead of making your pyramids going out of the dodecahedron, your pyramids went into the dodecahedron? What would the total angular deficit be?