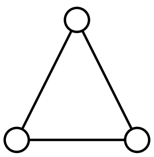
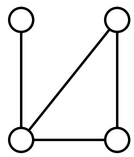
1. Graphs and Colorings

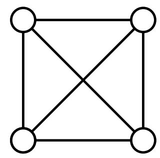
Definition. A *graph* is a collection of vertices, and edges between them. They are often represented by a drawing:



3 vertices
3 edges



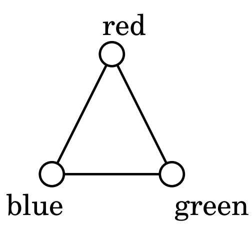
4 vertices
4 edges



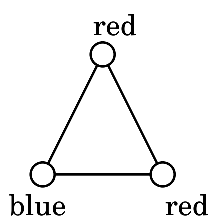
4 vertices
6 edges

A *graph coloring* assigns a color to each vertex, in a way so that no edge has both vertices the same color.

Example.



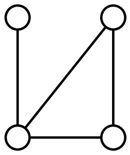
Good



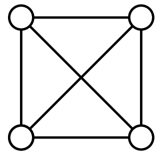
Bad

The figure on the left is a graph coloring, but the figure on the right is not. Why not?

Question. *Find a graph coloring of*



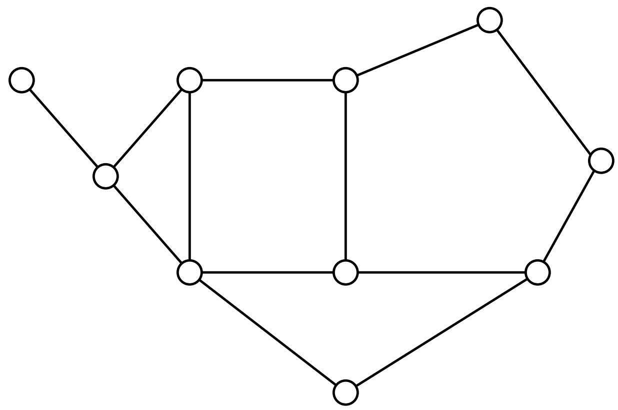
and



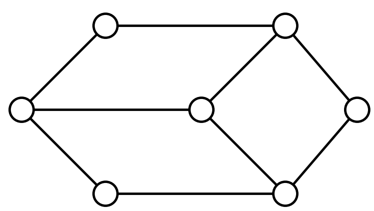
What is the least number of colors you can use?

2. More Colorings

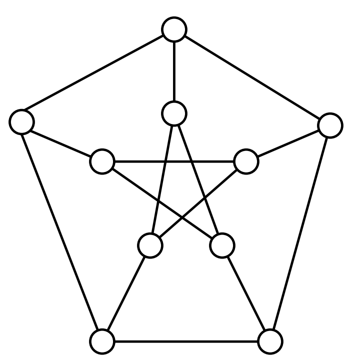
Question. *Color this graph. Try to use as few colors as possible.*



Question. *Color this graph. Can you do it with 2 colors?*



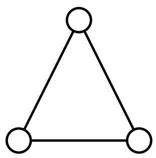
Question. *Color this graph. What is the least number of colors you can use?*



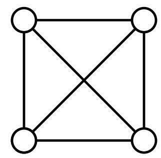
3. Complete Graphs

Definition. A *complete graph* has an edge between every 2 vertices. It's called complete, because you can't add any more edges.

The complete graph with n vertices is called K_n. So K_3 is



and K_4 is



Question. *Draw* K_5 *and* K_6.

Question. *What is the least number of colors to color* K_n*? Try it with* K_3, K_4, *and* K_5*!*

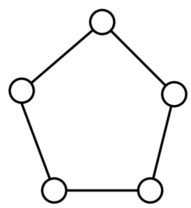
Question. K_n *has* n *vertices. How many edges does it have?*

4. Chromatic Number

Definition. The least number of colors needed to color a graph G is called the *chromatic number* of that graph.

Question. *What is the chromatic number of K_n?*

A cyclic graph can have its vertices arranged in a circle, with edges only on the outside. For example, the cyclic graph with 5 vertices is:



The cyclic graph with n vertices is called C_n.

Question. *What is the chromatic number of C_n? Try it with C_3, C_4, and C_5!*

Question. *C_n has n vertices. How many edges does it have?*

5. Graphs With No Odd Cycles

Notation. Sometimes we use the greek letter χ (chi) to represesent the chromatic number of a graph G, so that

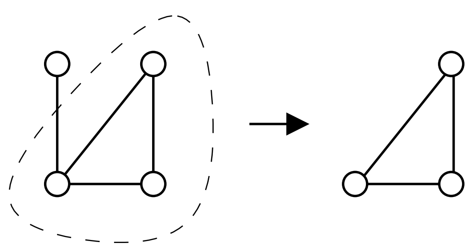
$$\chi(G) = \text{the chromatic number of } G.$$

You just showed that the chromatic number $\chi(C_n)$ of C_n is

$$\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases}.$$

Definition. An *induced subgraph* of a graph is a subset of vertices, with all the edges between those vertices that are present in the larger graph.

Example.



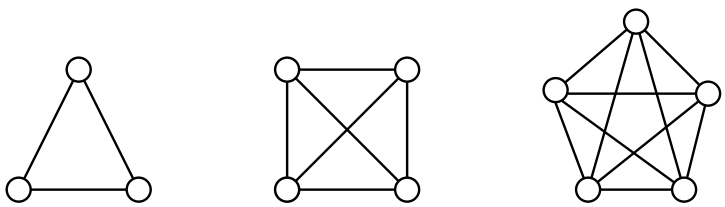
Question. *If a graph G has no induced subgraph which is an odd cycle, is the chromatic number $\chi(G) = 2$? Explain why, or give an example where this is not true.*

Definition. A graph with chromatic number 2 is called a *bipartite graph*.
This is because we can partition the vertices into two (bi) classes.

I guess you could call a graph with chromatic number 3 a *tripartite graph*, but for some reason, graph theorists don't usually do that.

6. Chromatic Number and Cliques

Remember that K_n is the complete graph on n vertices – the graph with n vertices, and all edges between them. You showed on Sheet 4 that the chromatic number of K_n is n.



Question. *Show that if G has an induced subgraph which is a complete graph on n vertices, then the chromatic number is at least n. I.e., $\chi(G) \geq n$.*

Definition. An induced subgraph which is complete is called a *clique*, since it's a small group where every vertex "talks to" every other vertex.

Question. *If the largest clique in G has n vertices, is the chromatic number necessarily equal to n? Explain why, or give an example where this is not true.*

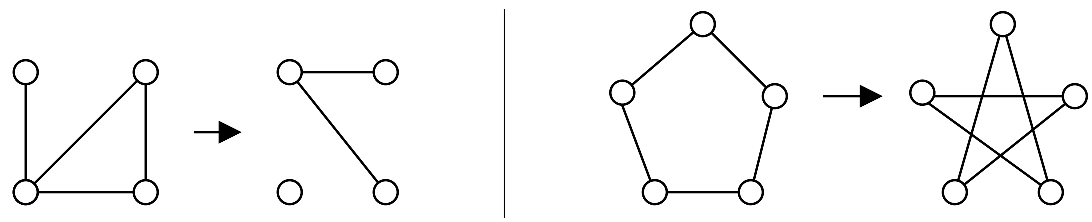
7. Perfect Graphs

Definition. A graph G where the chromatic number is equal to the size of the largest clique in G is called a *perfect graph*.

You probably showed on Sheet 6 that an odd cycle is not a perfect graph, since the largest clique in any cycle has 2 vertices (is an edge), but the chromatic number of an odd cycle is 3.

Definition. If G is any graph, then the *complement graph of* G is the graph with the same vertices, and an edge between two vertices if and only if there is no edge in G. This is best seen with some examples:

Examples.

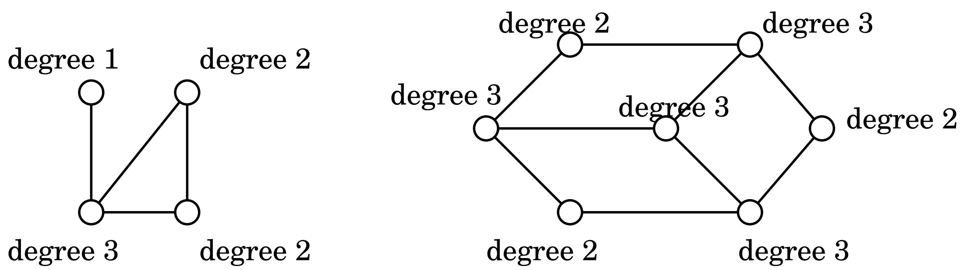


Question. *Draw the complement graph of* C_7*. What is the largest clique? Show that this graph is not perfect.*

It is a famous (and very difficult!) theorem, called the Strong Perfect Graph Theorem, that a graph is perfect if and only if no induced subgraph is either an odd cycle of length ≥ 5 or the complement graph of an odd cycle of length ≥ 5.

Definition. The *degree* of a vertex in a graph is the number of edges which have that vertex as an endpoint.

Examples.



Notice that in both of those examples, if you add up the degrees of all the vertices, you get an even number.

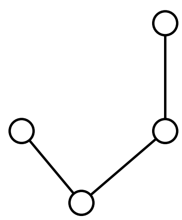
Question. *Does every vertex of odd degree have a neighbor of odd degree? Explain why this is always true, or give an example where it is not.*

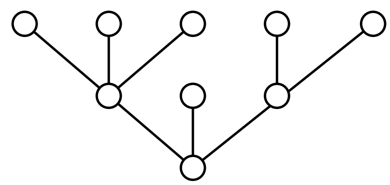
Question. *Is the sum of the degrees of all vertices in a graph always even? Explain why this is always true, or give an example where it is not.*

9. Trees

Definition. A *tree* is a connected graph which contains no induced cycles. A *leaf* is a vertex with degree 1, i.e., with only one edge.

Examples.





Question. *Does every tree have a leaf?*

Question. *What is the chromatic number of a tree?*

Question. *How many edges does a tree have?*

Question. *Do trees have to be perfect graphs? (Remember that a perfect graph has chromatic number equal to the size of its largest complete subgraph.)*

10. Constructing Examples

You'll have to work hard to answer these questions!

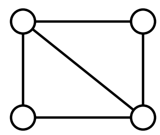
Question. *Find a graph* G *with no induced subgraph* K_4 *(i.e., no 4 vertices which are all adjacent to one another), such that* $\chi(G) = 4$.

Question. *Find a graph* G *with no induced triangle, such that* $\chi(G) = 4$.

Definition. A *chordal graph* is one that has no induced cycles of size greater than 3.
So a chordal graph can have triangles, but any larger cycle has some “chords” across it, which will break it up into triangles.

Examples.

(1) Any complete graph K_n.
(2) Any tree.
(3) The following graph, and many others like it:



Definition. A set of vertices is a *cut-set* for a graph G if removing the vertices disconnects G. That is, if removing the vertices leaves several subgraphs, with no edges in between them.

Example. Removing both vertices of the diagonal edge in Example (3) above disconnects the graph, so the diagonal edge is a cut-set for this graph.

Question. *Does the complete graph K_n have any cut-set? If so, describe the cut-set. If not, then explain why not.*

Question. *Does a tree have any cut-set? If so, describe the cut-set. If not, then explain why not.*

12. More Chordal Graphs

You'll have to work hard to answer these questions!

Question. *Show that if G is a chordal graph, but not a complete graph, then the smallest cut-set for G forms a complete subgraph.*

Question. *Using the fact from the above question, show that chordal graphs are perfect.*