Clock Arithmetic

Andrew Womack

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1 Clock Addition

Our first section of problems will focus on addition on a clock.

1.1 Standard Clock Problems

For these problems, we are using the standard clock with 12 points familiar to everyone.

1. If it is 9 and you get out of school in 4 hours, when do you get out of school?

2. If it is 11 and you have to finish your math homework in 126 hours, what hour will it be at that time?

3. How many hours do you need to go passed 5 to reach 12?

4. How many hours do you need to go passed 3 to reach 1?

5. How many hours do you need to go passed 6 to reach 2?

6. How many hours before 4 was it 8?

7. If the time reads 3 now, what will it read in 12 hours?

1.2 Exotic Clocks

1. Suppose that we have a clock with 24 points, if the current time is 10, how long do we have to wait until it is 1 again?

2. Suppose that we have a clock with 24 points, if it was 19 twelve hours ago, what is the current time?

3. Suppose that we have a clock with 31 points, what time is 17 hours before 14?

4. Suppose that you have a clock with 9 points, if the current time is 3 and you look at your watch ever 3 hours, what are the times you will see?

5. Suppose that you have a clock with 9 points, if the current time is 2 and you look at your watch every 2 hours, what times will you see?

6. Suppose that you have a clock with 9 points, if the current time is 4 and you look at your watch every 4 hours, what times will you see?

7. Suppose that you have a clock with 9 points, if the current time is 6 and you look at your watch every 6 hours, what times will you see?

8. Suppose that we have a clock with 45 points and that the clock currently reads 32. What will the clock read in 45 hours? What will the clock read in 0 hours?

1.3 General Ideas

1. Write out the addition table for clocks with 4, 5, 6, and 12 points.

2. What is 3 plus any number on a clock with 3 points? What is 6 plus any number on a clock with 6 points? How would you define 0 on these two clocks?

3. How would you define 0 on a clock with 23 points, on a clock with 24 points?

4. Change your addition tables using the defition of 0.

5. How would you define -5 on a clock with 16 points?

6. How would you define -5 on a clock with 6 points?

7. How would you define -5 on a clock with 2 points?

8. What are the numbers defined by 6, $6+6$, $6+6+6$, $6+6+6+6$, $6+6+6+6+6$ on a clock with 15 points? Such a set of numbers is called a circuit or cycle.

9. Give three examples of circuits for a clock with 6 points. These circuits are what we will call an additive sub-clock of a clock.

10. Can you give three examples of clocks for which the circuit given by 6 is an additive sub-clock?

11. Can you give three examples of clocks for which the circuit given by the time 4 is not an additive sub-clock (that is, where the circuit given by 4 is the entire clock)?

12. What is the additive sub-clock made by being able to move in units of 3 and 6 from 0 on a clock with 12? 4 and 6 on a clock with 12 points?

13. Find a clock in which movements by units of 4 and 6 from 0 generate the entire clock and one in which they do not.

2 Clock Multiplication

Now we will strive to define multiplication on clocks.

2.1 Standard Clock

All of the following questions are on a clock with 12 points.

1. How would you define 2×2?

2. How would you define 3×3?

3. How would you define 4×4?

4. Can you find two numbers so that their product is 12? If we have already decided that 12 behaves like 0 under addition, what does this mean? A number, say 2, such that there is another non-zero number, say 6, which multiplies it to 0 (which is because $2 \times 6 = 12 = 0$) is called a zero-divisor (in this example, 2 and 6 are both zero-divsors).

5. What do you make of the claim that $5 \times 5 = 1$? Can you find another number with this property? Another? Another (there are 4 total)? Any number with another number which multiplies by it to give 1 is

called a unit (in this example $5 \times 5 = 25 = 2 \times 12 + 1 = 2 \times 0 + 1 = 1$ and so 5 is a unit).

6. How many pairs of numbers (each no greater than 11 and no less than 1) can you find such that when you multiply them together you get 0 (that is, any multiple of 12)? Compute the greatest common divisor of each of these numbers and 12.

2.2 24 Point Clock

1. Find all of the units (those numbers with another number multiplying it to a multiple of 24 plus 1). An example is 5 because $5 \times 5 = 25 = 24 + 1 = 0 + 1$.

2. Compute the greatest common divisor of each unit and 24.

3. Is there something in common between the units?

4. Find all of the zero-divisors (those numbers with another number multiplying it to a multiple of 24). An example is 2 because $2 \times 12 = 24$.

5. Compute the greatest common divisor of each zero-divisor and 24.

6. Is there something in common between the zero-divisors?

7. Can you tell me what distinguishes the zero-divisors from the units?

2.3 General Ideas

1. Can you find a clock on which 5 is a unit but 5×5 is not 1?

2. Can you find a clock that has all non-zero numbers as units?

3. Write the multiplication tables for clocks with 4, 5, 6, and 12 points.

4. It is interesting that $2 \times 2 = 1$ on a clock with 3 points and that $2 \times 2 \times 2 = 1$ on a clock with 7 points. How many times must I multiply 4 by itself to get 1 on a clock with 9 points?

5. How many times must I multiply 7 by itself to get 1 on a clock with 13 points?

6. How many times must I multiply 5 by itself to get 1 on a clock with 17 points?

7. I give you a clock with 5 points. Find each unit. Multiply each unit by itself as many times as you need to in order to get 1. For example, 2 is a unit and $2 \times 2 = 4$, $2 \times 2 \times 2 = 8 = 5 + 3 = 0 + 3 = 3$, and $2 \times 2 \times 2 \times 2 = 3 \times 2 = 6 = 5 + 1 = 0 + 1 = 1$. We then say that the order of 2 is four on this clock.

8. Consider a clock with 9 points. How many units are there? What are the orders of the units? What relationship is there between the orders of the units and the number of units?

9. Consider a clock with 12 points. How many units are there? What are the orders of the units? What relationship is there between the orders of the units and the number of units?

10. Consider a clock with 15 points. How many units are there? What are the orders of the units? What relationship is there between the orders of the units and the number of units?

11. Suppose that we have a clock with 5 points. Show that any unit multiplied by itself 4 times gives 1.

12. On a clock with 6 points, show that any unit multiplied by itself 2 times gives 1.

13. Do you think that given any clock you will find a unit with order equal to the number of units?

14. On a clock with 18 points, what are the numbers generated by multiplying 1 by powers of 5? Is this then entire set of units?

15. On a clock with 16 points, is there a unit with order 8? Can you generate all of the units using powers of 3 and 5?

3 General Theory

3.1 Fermat's Theorem

Pierre de Fermat proved long ago (1640) that if you take any integer a which is not a multiple of p and look at $a^{p-1}-1$ for any prime p, something amazing happens. Let us try to figure out what that is.

1. On a clock with 3 points, what is 2^2, what is 4^2?

2. On a clock with 5 points, what is 2^4, what is 3^4, what is 4^4?

3. On a clock with 7 points, what is 2^6, what is 3^6, what is 4^6, what is 5^6, what is 6^6?

4. What do you think we will get on a clock with 31 points if we compute 15^{30}?

5. Can you tell me what Fermat's amazing discovery was?

3.2 Euler's Theorem

This is a super version of Fermat's Theorem. Leonhard Euler lived in the century after Pierre de Fermat. His early work followed some of Fermat's work and greatly expanded it. One of Euler's greatest contributions is closesly related to clock arithmetic, specifically clock multiplication. In particular, the set of units of a clock form a special mathematical object. Euler set out to count the number of units on a given clock. His success in this endeavor led to quite an amazing generalization of Fermat's Theorem. We will try to count the number of units, and then try to discover Euler's Theorem for ourselves.

1. Take a clock with 4 points. How many units are there? Raise each unit to that power, what do you get?

2. Take a clock with 6 points. How many units are there? Raise each unit to that power, what do you get?

3. Take a clock with 8 points. How many units are there? Raise each unit to that power, what do you get?

4. Take a clock with 9 points. How many units are there? Raise each unit to that power, what do you get?

5. Can you tell me how many units there are on a clock with 30 points? What is 7 to that power?

6. Can you make a guess at what Euler's Theorem was?

3.3 Euler's ϕ Function

For those of you that are interested, Euler defined a function that counts the number of units on a given clock. He called this function ϕ. Examples: $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, $\phi(6) = 2$, $\phi(7) = 6$, and $\phi(8) = 4$. There is a nice general formula that you might be able to discover for yourself. Here are the steps (try working each our for some small numbers, like $p = 2, 3$, $n = 1, 2, 3, 4$, and a and b less than 9.

1. Show that if p is a prime, then $\phi(p) = p - 1$.
2. Show that if p is a prime, then $\phi(p^2) = p \times (p - 1)$.
3. Show that if p is a prime, then $\phi(p^3) = p^2 \times (p - 1)$.
4. Show that if p is a prime, then $\phi(p^n) = p^{n-1} \times (p - 1)$.
5. Show that if the greatest common divisor of a and b is 1, then $\phi(a \times b) = \phi(a) \times \phi(b)$.
6. Combine what you have learned from the above steps. Tryt computing $\phi(648)$.