

Mathematical Battle

Math Circle - Summer 2009

Rules

- You will be divided into two teams. You will be given approximately two hours to work on the problems as a team.
- There will be three judges who decide on scores. The decisions of the judges are final.
- Each team selects a captain who serves as spokesperson for the team and also participates in the captain's contest.
- The battle begins with a short question to be answered on the spot. The victor decides whether to begin with the right to challenge or to pass this right to the other team.
- At each stage of the battle, the team with the right to challenge chooses a problem from those that have yet to be presented and challenges the opposing team to present a solution.
- When challenged, the opposing team may choose to accept the challenge, in which case they present a solution. They may also opt to return the challenge, in which case the original team must attempt to present a solution.
- The team presenting a solution nominates one member who has not yet discussed a problem to provide an explanation. This person has up to five minutes to present as complete a solution to the problem as they are able. Drawing and writing equations is included in this five minutes. The presenter may briefly discuss the problem with their team before stepping to the board, but they may not consult with their team while describing their solution.
- The other team then nominates one member who has not yet discussed a problem to respond to the solution just presented. This person has up to three minutes to point out any flaws or omissions or even supply an alternate solution. The respondent can discuss their rebuttal briefly with their team but may not consult their team while speaking.
- After the presentation and rebuttal, the judges may pose questions to one or both of the speakers.
- The judges then award the points available among the three parties involved: the presenting team, the rebutting team and the judges. Each problem is initially worth 7 points. However, should a team return a challenge, the value of the problem increases to 10 points.
- With one exception, the right to challenge then passes to the next team.

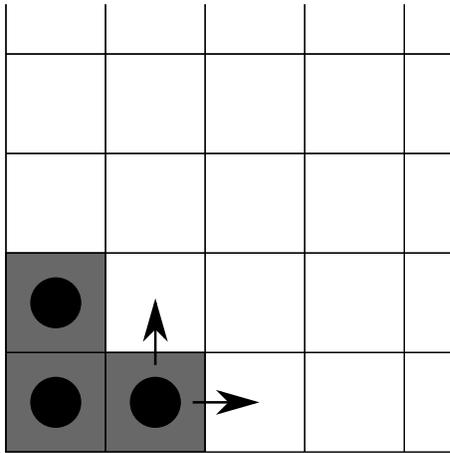
Exception: when a team returns a challenge and the original team is unable to make significant progress toward a solution (defined as receiving 3 or fewer points), the right to challenge remains with the original team.

Problems

1. Bill takes three minutes to descent the down escalator standing still. Ted walks down the same escalator in one minute, taking 120 steps in transit. If each step is 20cm high, then what is the overall height of the escalator?
2. Drew draws diagonals within some of the unit squares of an 8×8 chessboard so that none of the diagonals intersect, even at their endpoints. Find, with proof, the maximum number of diagonals that Drew can draw.
3. A seventeen digit number is chosen, its digits are reversed to form a new number. These two numbers are added together. Prove that their sum contains an even digit.
4. Given that p , $p + 10$ and $p + 14$ are all prime numbers, find all possible p .
5. Given 12 integers, prove that 2 of these integers can be selected such that their difference is divisible by 11.
6. A regular tetrahedron is a polyhedron with four faces, each an equilateral triangles. Use the center of each triangle as a vertex for another, smaller, tetrahedron.
If the larger tetrahedron has volume of 54, what is the volume of the smaller tetrahedron?
7. The boxes of an $n \times (n + 1)$ table are filled with integers. Prove that one can cross out several columns (or none, but not all of them) so that after this crossing out, the sum of the numbers in each row is even.
8. The numbers p and $2^p + p^2$ are both prime. Find all possible p .
9. Prove that a natural number written with one 1, two 2's, three 3's, \dots , nine 9's cannot be a perfect square.
10. Can a convex 13-gon be dissected into parallelograms? Prove your answer is correct.

11. Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to one of its diagonals. Prove that at least one of the checkers is positioned on the diagonal.
12. Prove that a convex polygon cannot have more than three acute angles.
13. Consider the infinite grid with the “clones” (dots) trapped in the “prison” in the lower left as pictured. Each of the clones can clone itself and the two new clones move to the cell above and to the right (as shown by the arrow). Thus, each cloning move increases the total number of clones by one.

By performing a sequence of these cloning moves, can you free all the clones from the prison? Justify your answer.



14. None of the numbers a, b, c, d, e, f are zero. Prove that there are both positive and negative numbers among the numbers

$$ab, cd, ef, -ac, -be, -df$$