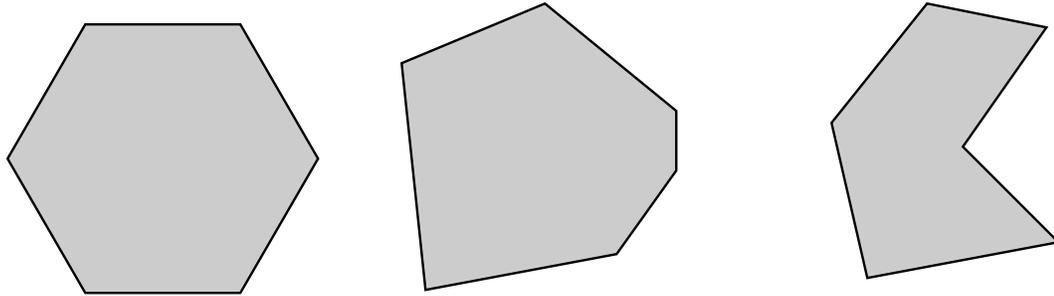


1. CONVEX POLYGONS

Definition. A shape D in the plane is *convex* if every line drawn between two points in D is entirely inside D .



Convex 6-gon

Another convex 6-gon

Not convex

Question. *Why is the third shape not convex?*

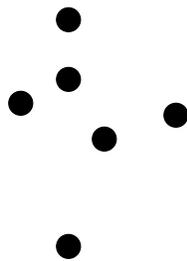
Definition. A shape D is a *polygon* if it satisfies can be described either as:

- (1) The smallest convex shape enclosing a finite set of points in the plane. Or:
- (2) The closed shape bounded by a finite number of lines.

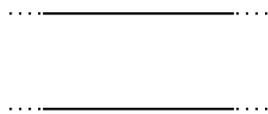
A polygon with n sides is called an n -gon. Some have special names: a 3-gon is also called a triangle, a 4-gon a square, a 5-gon a pentagon, etc.

Question. *Give both styles of description for the two convex polygons above.*

Question. *Which of the following two gives a polygon? If a polygon, how many sides does it have?*



Smallest convex shape enclosing the above 6 points



The shape bounded by the above two lines

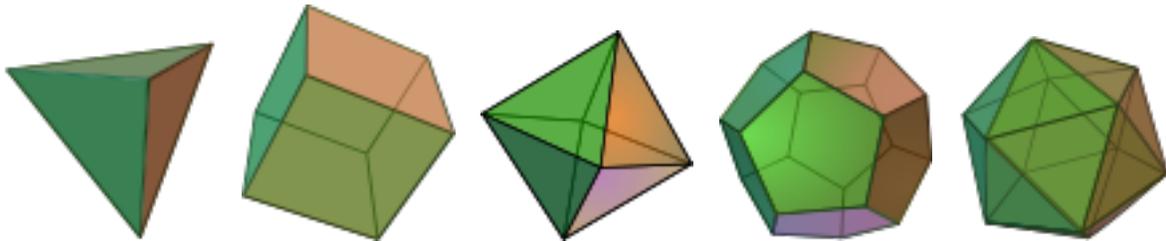
2. CONVEX POLYHEDRA

Definition. A shape is a *convex polyhedron* if it can be described either as:

- (1) The smallest convex shape enclosing a finite set of points in 3 dimensions. Or:
- (2) The closed shape bounded by a finite number of planes.

The first description is called the *convex hull* of the points.

Example. The platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. Pictured below – see also the 3d models.



(Graphics source: Wikipedia)

Example. Describing the tetrahedron:

- (1) Take any 4 points in 3-space that don't all lie on any plane. Their convex hull is a tetrahedron.
- (2) The region bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ is a tetrahedron.

Question. Describe the cube both as a convex hull of finitely many points and as the region bounded by planes.

Question. If convex polygons are 2-polytopes, and convex polyhedra are 3-polytopes, what do you think the definition of a 4-polytope should be?

3. VERTICES, EDGES, AND FACES

Polyhedron have vertices, edges, and faces, which I take as understood.

Each face of a polyhedron is a polygon.

If two polyhedra have vertices, edges, and faces that “fit together in the same way”, then we regard them as being the same. For example, the cube is the same as a box, a regular tetrahedron is the same as a tall tetrahedron, etc.

Question. *Given any polyhedron, describe how to:*

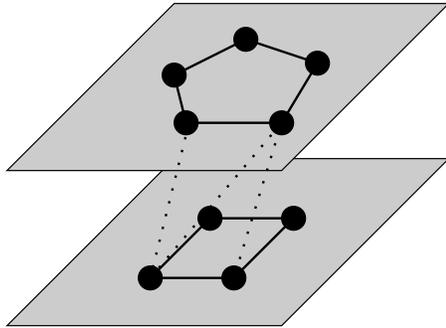
- (1) *Use its vertices to express it as a convex hull.*
- (2) *Use its faces to express it as a closed shape bounded by a finite number of planes.*

Question. *Give an example of a polyhedron which is not a platonic solid.*

Can you find a polyhedron that has both a 4-gon and a 5-gon as faces?

How about a polyhedron that has 8 faces, but is not the octahedron?

4. GRAPHS



Did you find a polyhedron with both a square and pentagon as faces? One way to do it is as follows: put 4 points in a square in one plane, and 5 points in a pentagon in another plane. Take the convex hull of these 9 points.

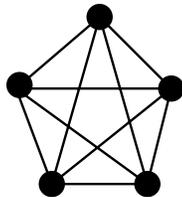
A *graph* is a collection of vertices, and edges between them. They are often represented by a drawing. For example:



A graph is *planar* if it can be drawn in the plane without any of its edges crossing. (Such a drawing is called a *planar representation*.)

Question. *The two graphs above are both planar graphs, although the drawings given are not planar representations. Draw a planar representation of each.*

Question. *The complete graph on 5 vertices, pictured below, cannot be drawn in the plane. Why not?*

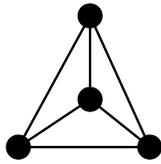


5. GRAPHS OF POLYHEDRA

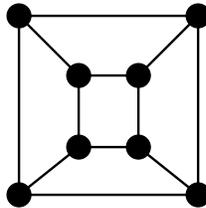
Definition. Any polyhedron has a graph associated with it, where the vertices of the graph are the vertices of the polyhedron, and the edges of the graph are the edges of the polyhedron. (Thus, the graph is obtained from the polyhedron by ignoring the faces.)

Since in passing to this graph you remove the “body” of the polyhedron, leaving only the “bones”, it is sometimes called the *skeleton graph* or *1-skeleton* of the polyhedron.

Example. Skeleton graphs



Tetrahedron

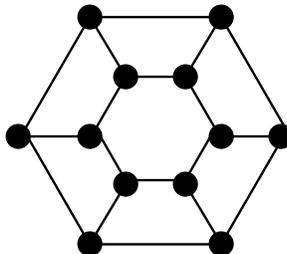
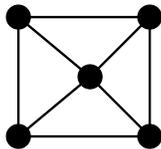


Cube

The skeleton graph of a polyhedron is always planar. To get a planar representation, think of putting your eye close enough to a face of the polyhedron that you can see the entire polyhedron. The edges will be straight, and not cross each other – draw them in the plane.

Question. Draw a planar representation of the skeleton graph of the octahedron.

Question. What polyhedra do the following graphs represent?



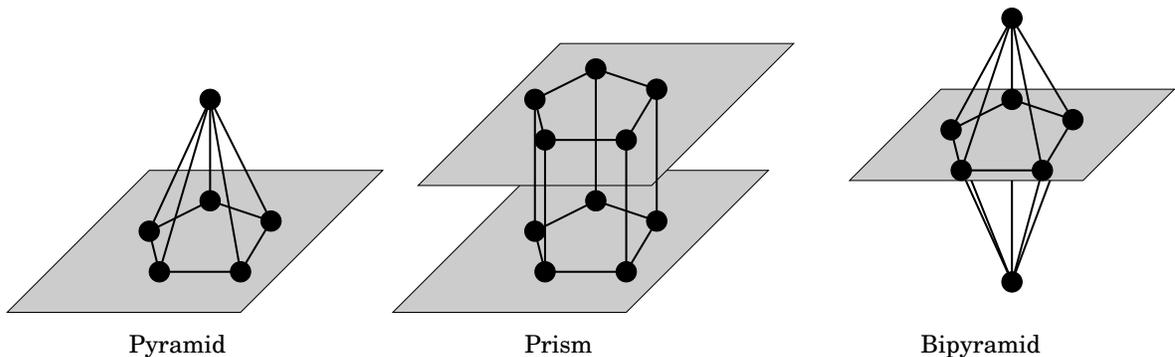
6. LOTS OF EXAMPLES

Unlike the situation with polygons, where every polygon is an n -gon, there are too many polyhedra for a clean classification. But there are some interesting infinite families of polyhedra.

Definition. The *pyramid over the n -gon* is obtained as follows: take the n -gon in a plane, put a single point above it, and take the convex hull.

Definition. The *prism over the n -gon* is obtained as follows: take the convex hull of an n -gon together with a translated copy, as pictured.

Definition. The *bipyramid over the n -gon* is obtained as follows: take the n -gon in a plane, put one point above it and another below it, and take the convex hull.



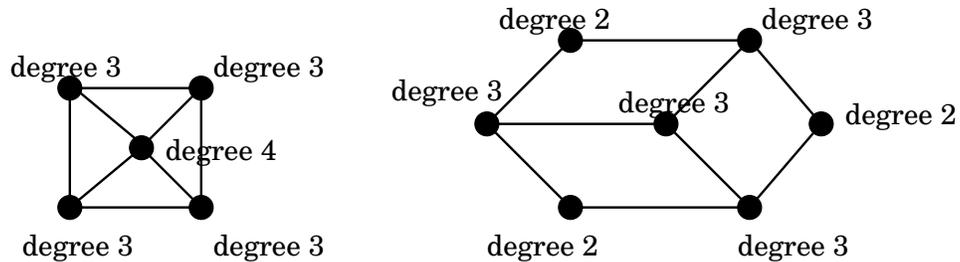
Question. Describe each of the tetrahedron, the cube, and the octahedron as a pyramid, prism, or bipyramid.

Question. What are the skeleton graphs of the pyramid over the n -gon and the prism over the n -gon?

7. DEGREE

Definition. The *degree* of a vertex in a graph is the number of edges which have that vertex as an endpoint.

Examples.



Notice that in both of those examples, if you add up the degrees of all the vertices, you get an even number.

Question. What is the smallest degree a vertex can have in the skeleton graph of a polyhedron. Give an example, and explain why it can't be smaller.

Question. Is the sum of the degrees of all vertices in a graph always even? What is the sum of the degrees?

8. A VERTICES-EDGES RELATION

You probably showed on Sheet 7 that the sum of the degrees of vertices of any graph is even.

In fact, the sum of the degrees of vertices is always twice the number of edges, since every vertex v has $\text{degree}(v)$ edges adjacent to it, and every edge connects 2 vertices.

You also probably showed that the degree of a vertex in the skeleton graph of a polyhedron is at least 3.

Question. *Suppose a polyhedron has a number V of vertices, E of edges, and F of faces. Suppose furthermore that every vertex of the polyhedron has degree exactly 3. What is the relationship between V and E ?*

Remark. A polyhedron where every vertex has degree 3 is called a *simple polyhedron*. A polyhedron where every face has 3 edges is called a *simplicial polyhedron*.

From here on V , E , and F will always be the number of vertices, edges, and faces of a polyhedron.

Question. *If a polyhedron has vertices of any degree (each degree at least 3, of course!), what is the relationship between V and E ?*

9. VERTEX, EDGE, AND FACE INEQUALITIES

Remember that V , E , and F are always the number of vertices, edges, and faces of a polyhedron.

You probably showed on Sheet 8 that $3V \leq 2E$.

A similar argument will show that $3F \leq 2E$. Briefly sketched: any face is an n -gon for $n \geq 3$, hence is adjacent to 3 edges, and the sum of “face-adjacencies” is $2E$, so $3F \leq 2E$.

Question. *Is there a polyhedron where $3V = 2E = 3F$?*

Question. *What about a polyhedron where $(V, E, F) = (6, 12, 8)$?*

Question. *What about a polyhedron where $(V, E, F) = (8, 12, 10)$?*

10. EULER'S RELATIONSHIP

On Sheet 9, you probably a polyhedron where $3V = 2E = 3F$: the tetrahedron. That's the only example.

We've seen that if V , E , F are the vertex, edge, face numbers of a polyhedron, then

$$(1) 3V \leq 2E,$$

$$(2) 3F \leq 2E.$$

The other famous relationship between V , E , and F is one that you may have seen before, due to Leonhard Euler:

$$V - E + F = 2.$$

Question. (Only if you've seen Euler's formula before)

Recall a proof of Euler's formula: By looking at the planar representation of the skeleton graph of a polyhedron, show that Euler's formula holds. Match edges with the faces until you have no faces left, then match edges with vertices.

Question. (If you've never seen Euler's formula)

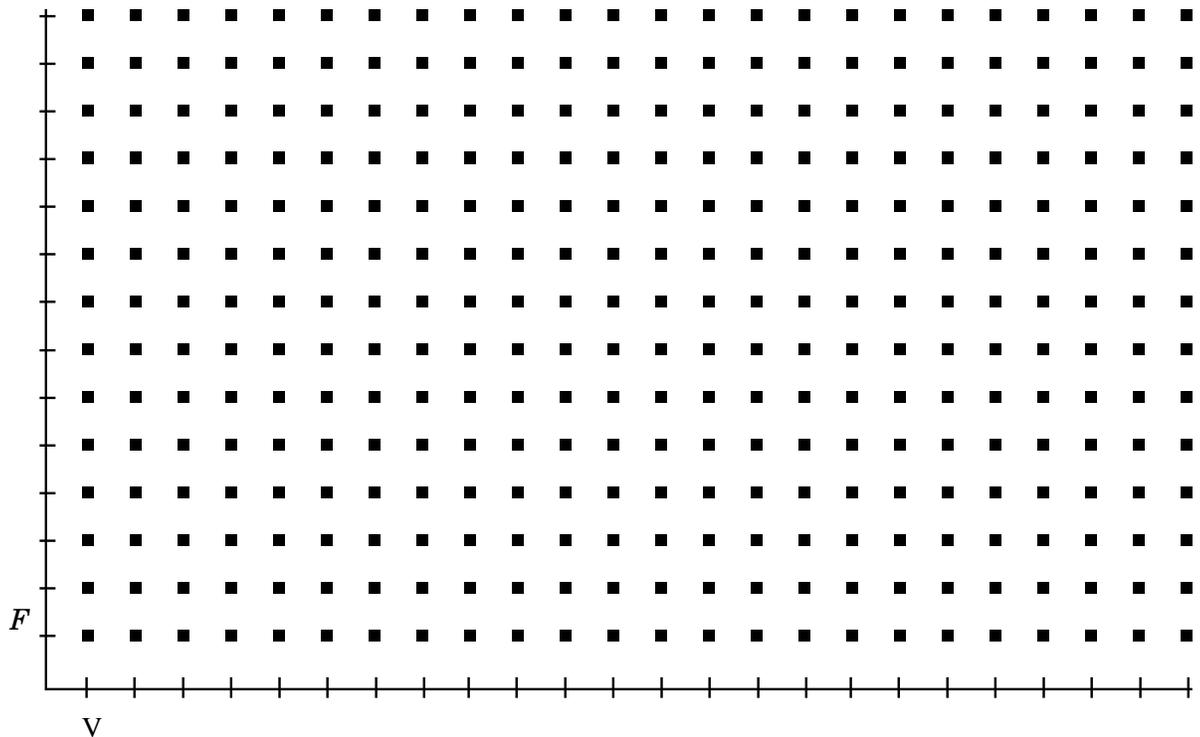
Show that Euler's formula holds for the tetrahedron, cube, and octahedron.

11. V AND F DETERMINE E

Using Euler's formula, you can eliminate E from any equations or inequalities involving V , E , and F .

Question. Write the relations $3V \leq 2E$ and $3F \leq 2E$ in terms of V and F alone.

On the axis below, indicate the (V, F) pairs that could possibly be the vertex and face numbers of a polyhedron.



Question. If n is any natural number ≥ 4 , find a polyhedron with exactly n vertices and exactly n faces.

12. REALIZING (V, F)

On Sheet 11, you probably showed that the (V, F) pairs that could possibly be the vertex, face numbers of a polyhedron must be integer points in the following cone:

$$V \leq 2F - 2 \text{ and } F \leq 2V - 2, \text{ with } V, F \geq 4.$$

If a (V, F) point is outside this cone, then there is no polyhedron with that many vertices and faces.

You've shown that every integer point on the line $V = F$ (for $V \geq 4$) can be realized by an actual polyhedron. What other points in the cone can be so realized?

Question. *Find a polyhedron with $(V, F) = (20, 12)$. What are the degrees of the vertices and how many edges does each face have?*

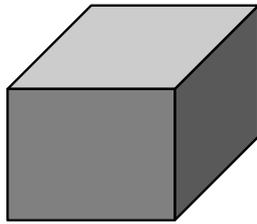
Question. *Find a polyhedron with $(V, F) = (6, 10)$. What are the degrees of the vertices and how many edges does each face have?*

13. NEW POLYHEDRA FROM OLD

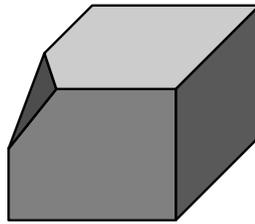
Let's mention two operations we can perform on a polyhedron to get a new polyhedron with more vertices and faces.

- (1) We can “cut off” or “truncate” a vertex, by adding a new plane to the representation of the polyhedron as the region bounded by a finite number of planes.
- (2) We can “cone” over a particular face, by adding a new point just over that face. (The new point needs to be close enough to maintain convexity!)

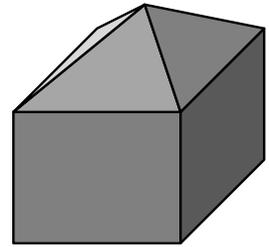
A picture is worth 1000 words:



The cube



The cube with its front upper left vertex truncated



The cube with a cone over its top face

Question. *What does coning over a face do to the skeleton graph of a polyhedron? What about truncation at a vertex? Draw some examples!*

Question. *How does coning over a face change V , E , and F ? What about truncation at a vertex?*

Question. *How does coning over a face change the degrees of the vertices of a polyhedron? How many edges do the added faces have?*

14. ONE MORE QUESTION

Question. *Using coning and truncation, show that every (V, F) satisfying $V \leq 2F - 2$, $F \leq 2V - 2$, and $V, F \geq 4$ is the number of vertices, faces of a polyhedron.*

You'll have to work hard to get this one!

It might be easier to show it for $V \leq F \leq 2V - 2$.