

Math Circle: Recursion and Induction

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1 Recursion

What can we compute, using only simple formulas and rules that everyone can understand?

1. Let us use \mathbf{N} to denote the set of counting numbers $\{1, 2, 3, \dots\}$. It is impossible to write down the complete list, but we can imagine it. We can also imagine picking a number from the list and, keeping it secret for a while, using the letter n to name it.

For any such n , the next number will be $n + 1$. Other arithmetic rules apply: twice the number is $2n$, ten times the number is $10n$, the square of the number n^2 , and so on.

Some operations will sometimes take us out of \mathbf{N} . For example, $n - 1$ is not a counting number (in \mathbf{N}) if $n = 1$, since $1 - 1 = 0$ is not in \mathbf{N} . But $n - 1$ is the previous counting number for all $n \geq 2$.

2. * For what counting numbers n is $n/2$ a counting number?
3. * For what counting numbers n is $-n$ a counting number?
4. * For what counting numbers n is $(n + 1)/2$ a counting number?
5. ** For what counting numbers n is $n(n + 1)/2$ a counting number?

6. How would we generate a list of all the counting numbers? Easy — start with 1, and then keep adding 1's forever:

$$1, 2, 3, 4, 5, 6, 7, \dots$$

This rule may be expressed as two steps:

- A. Pick the initial number 1
- B. The next number equals the current number plus one.

7. * What would the two steps be for generating all the odd counting numbers?

8. * What would the two steps be for generating all the even counting numbers?

9. A *function* on \mathbf{N} is a rule that gives a number for each counting number. Give it a name, like “ f ,” and write $f(n)$ for the number it produces at input n .

A function can also be specified with an arithmetic rule involving the input n . For example, the next-counting-number function is $f(n) = n + 1$. The doubled-number function is $f(n) = 2n$.

10. * Find the first four values $f(1), f(2), f(3), f(4)$ of the next-counting-number function.

11. * Find the values for the doubled-number function $f(n) = 2n$ for $n = 111$ through $n = 115$.

12. * Suppose $f(n)$ is the sum of all the counting numbers from 1 to n . Is $f(n)$ always in \mathbf{N} ? What are $f(2), f(3), f(4)$?

13. We can imagine describing a function f using a table, such as:

n	1	2	3	4	...
$f(n)$	1	3	6	10	...

This works well if we only need the first few values $f(1), f(2), \dots$, and has the advantage of no arithmetic.

14. * For the f given above, what is $f(2)$? $f(3)$? $f(3) - f(2)$? $f(n + 1) - f(n)$ for $n = 1, 2, 3$?

15. We can describe a function in pieces. For example, consider the parity-label function:

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

This has the table

n	1	2	3	4	...
$f(n)$	1	2	1	2	...

16. * Write the table for the parity-label function's values from $n = 2012$ to $n = 2015$.

17. The pieces could be formulas. For example, consider the function:

$$f(n) = \begin{cases} 3n + 1, & \text{if } n \text{ is odd,} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

18. * Write the table for this function's values from $n = 1$ to $n = 8$.

19. Consider the rules we can use to fill out a table from left to right. After n columns are filled, we will have listed the successive counting numbers $1, 2, \dots, n$ and also the function values $f(1), f(2), \dots, f(n)$. Thus we can apply a rule that uses the previous value $f(n)$ to compute the next value $f(n + 1)$.

It is necessary to start things off by defining $f(1)$. Thus the two steps needed to define a function this way are:

- A. Pick the initial value $f(1)$
- B. Compute $f(n + 1)$ with a formula using $f(1), \dots, f(n)$.

Functions f defined this way are called *recursive*. Filling a table with their values is called *recursion*. Step A is called the *initial condition*, while step B is called the *recurrence equation*.

20. * For an example, consider

- A: $f(1) = 1$
- B: $f(n + 1) = 2f(n)$

That has the table form

n	1	2	3	4	5	6	7	8	...
$f(n)$	1	2	4	8	16				...

What are the next values $f(6), f(7), f(8)$ in the table?

21. * Write the table form for the rule

- A: $f(1) = 2$
- B: $f(n + 1) = f(n)^2$

for $n = 1, 2, 3, 4, 5$.

22. A recurrence equation may also be described in pieces. One example is

$$f(n+1) = \begin{cases} 3f(n) + 1, & \text{if } f(n) \text{ is odd,} \\ f(n)/2, & \text{if } f(n) \text{ is even.} \end{cases}$$

This used to generate the *Collatz sequence* from an initial counting number x :

A: $f(1) = x$

B: $f(n+1) = \begin{cases} 3f(n) + 1, & \text{if } f(n) \text{ is odd,} \\ f(n)/2, & \text{if } f(n) \text{ is even.} \end{cases}$

For example, the Collatz sequence starting from $f(1) = 8$ has the following table:

n	1	2	3	4	5	6	7	...
$f(n)$	8	4	2	1	4	2	1	...

23. If the Collatz sequence starting from x ever hits 1, it repeats $(4, 2, 1)$ endlessly thereafter:

n	1	2	3	4	5	6	7	...
$f(n)$	1	4	2	1	4	2	1	...

The *Collatz conjecture*, stated in 1937 and still unproven, is that the Collatz sequence starting from any x in \mathbf{N} will eventually hit 1.

24. * Write the Collatz sequence starting from 3.

25. * Write the Collatz sequence starting from 5.

26. * Write the Collatz sequence starting from 7.

27. Recursive function rules may require pre-defining the first two or more values $f(1), f(2), \dots$, as in the example

A1: $f(1) = 1$

A2: $f(2) = 1$

B: $f(n + 2) = f(n + 1) + f(n)$

which generates the *Fibonacci sequence*.

28. * Write the table for the first seven values of the Fibonacci sequence.

29. Sometimes a recursion can be *solved*, or replaced with a formula for $f(n + 1)$ that does not use previous values $f(n), \dots, f(1)$. For example,

A: $f(1) = 1$

B: $f(n + 1) = 2f(n)$

has the solution $f(n) = 2^{n-1}$. Likewise,

A: $f(1) = 2$

B: $f(n + 1) = f(n)^2$

has the solution $f(n) = 2^{2^{n-1}}$.

The Fibonacci sequence solution is shown at the end of this section.

We get such formulas by conjecturing the pattern from the first few values, then proving it with the method in the section on Induction.

30. * Try to solve the recursion

A: $f(1) = 1$

B: $f(n + 1) = 3f(n)$

31. * See if you can solve the recursion

A: $f(1) = 1$

B: $f(n + 1) = 2f(n) - 1$

32. Some recursions are so common that special names exist for their solutions, such as

A: $f(1) = 1$

B: $f(n + 1) = (n + 1)f(n)$

whose solution is called the *factorial function*, $f(n) = n! = n(n - 1) \cdots (2)(1)$.

33. * Evaluate the factorial function for $n = 2, 3, 4, 5$.

34. * The *double factorial function* $f(n) = n!!$ is defined by the recursion

A1: $f(1) = 1$

A2: $f(2) = 2$

B: $f(n + 2) = (n + 2)f(n)$

Write the table for $f(n)$ for $n = 3, 4, 5, 6, 7, 8$.

35. It is easy to write down simple recursive function rules that do not have simple solutions, or even known solutions. For example,

A: $f(1) = 1$

B: $f(n + 1) = 1 + f(n)^2$

has no formula solution. One can, however, get some estimates with a formula like $f(n) \geq 2^{2^{n-1}}$ by comparison with a solvable recurrence like $f(n + 1) = f(n)^2$.

36. * Fill out the following table for the recurrences $f(n + 1) = f(n)^2$ and $g(n + 1) = 1 + g(n)^2$:

n	1	2	3	4	5
$f(n)$	1				
$g(n)$	1				
$f(n)$	2				
$g(n)$	2				

37. Some special cases *do* have reasonable solution methods. For example, given fixed numbers c and r , the recursion

A: $f(1) = c$

B: $f(n + 1) = rf(n)$

has the solution $f(n) = cr^{n-1}$. This may be proved by induction, to be discussed in detail later. For now, note that the solution formula is true for $n = 1$, since $cr^{1-1} = c$, and that the formula for $n + 1$ derives from the formula for n by substitution and the law of exponents:

$$f(n + 1) = rf(n) = r(cr^{n-1}) = cr^{1+n-1} = cr^{(n+1)-1}$$

38. * Evaluate $f(4)$ for the function above with $c = 3$, $r = 2$.

39. * Find numbers c, r such that the solution is $f(n) = 1$ for all n in \mathbf{N} .

40. Another special case chooses three fixed numbers c, r, s in the recursion

A: $f(1) = c$

B: $f(n + 1) = rf(n) + s$

This has the solution $f(n) = cr^{n-1} + s[r^{n-2} + \dots + r + 1]$, where the part in square brackets appears only for $n \geq 2$. This solution may be guessed by repeated substitution and then proved by induction: the solution formula is true for $n = 1$, since $cr^{1-1} = c$, and the formula for $n + 1$ derives from the formula for n by some algebra:

$$f(n+1) = rf(n)+s = r(cr^{n-1}+s[r^{n-2}+\dots+r+1])+s = cr^{(n+1)-1}+s[r^{(n+1)-2}+\dots+r+1]$$

41. * Keeping the c, r, s form, write out the formulas for $f(n)$ for $n = 1, 2, 3, 4$.

42. There is also an algebra method to solve *linear* recurrences, which are of the form

$$\text{A1: } f(1) = c_1$$

⋮

$$\text{Am: } f(m) = c_m$$

$$\text{B: } f(n+m) = d_m f(n+m-1) + \cdots + d_1 f(n)$$

where m, c_1, \dots, c_m and d_1, \dots, d_m are fixed numbers.

The Fibonacci sequence comes from such a linear recurrence, with $m = 2$, $c_1 = c_2 = 1$, and $d_1 = d_2 = 1$. All linear recurrences are solved by finding the m roots of the *characteristic polynomial*

$$p(x) = x^m - d_m x^{m-1} - \cdots - d_1 x^0$$

In the generic case where no two of the roots are equal, call them $\{r_1, \dots, r_m\}$. Then the solution is of the form

$$f(n) = a_1 r_1^n + a_2 r_2^n + \cdots + a_m r_m^n,$$

where $\{a_1, \dots, a_m\}$ are fixed numbers determined by the initial values $\{c_1, \dots, c_m\}$.

For the Fibonacci sequence, the characteristic polynomial is quadratic with distinct roots,

$$p(x) = x^2 - x - 1; \quad r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2},$$

and the numbers a_1, a_2 solve the linear system of equations

$$\begin{aligned} a_1 r_1 + a_2 r_2 &= c_1 = 1 \\ a_1 r_1^2 + a_2 r_2^2 &= c_2 = 1, \end{aligned}$$

which after some algebra gives $a_1 = 1/\sqrt{5}$ and $a_2 = -1/\sqrt{5}$.

43. ** Prove that for any n in \mathbf{N} , the quantity

$$\frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

is a counting number. Hint: this is Fibonacci sequence member n .

2 Induction

If you have a guess for the formula that solves a recursion, then there is a way to prove that it always works.

1. Use \mathbf{N} to denote the set of numbers $\{1, 2, 3, \dots\}$. We'll use them to label lists of statements that we want to prove. For example:

Statement 1: The sum of the first 1 odd numbers is 1

Statement 2: The sum of the first 2 odd numbers is $1 + 3 = 4$

Statement 3: The sum of the first 3 odd numbers is $1 + 3 + 5 = 9$

...

Rather than "Statement 1," "Statement 2," "Statement 3," it is easier to write $S(1)$, $S(2)$, $S(3)$, and so on.

2. * The statements above are all true. What would a true $S(5)$ say?

3. Three dots (...) is called an *ellipsis*, which means "omission." It replaces left-out terms in a long statement when the pattern is easy to see. For example, the sum of the first 100 odd numbers may be written

$$1 + 3 + 5 + \dots + 199 \quad (= 10000, \text{ to be shown later})$$

4. * How would you write the sum of the first 100 even numbers using ellipsis?

5. * What is the value of that sum of the first 100 even numbers? Hint: each even number is one more than an odd number.

6. Some patterns are more obvious than others. For example, “1,4,9,16,25,...” is readily identified from the first 5 terms as the squares of numbers, so the next term is 36.

7. * Consider the pattern

$$1, 2, 4, 8, 16, 32, \dots$$

What should the next number be?

8. * Some patterns are less obvious, such as

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

Each term after the second depends on the two previous terms. What should the next number be?

9. ** Some patterns can be very obscure, such as

$$3, 7, 15, 1, 292, 1, 1, 1, \dots$$

These numbers depend on π . What should the next number be? Hint: use a search engine.

10. A *variable* like the letter n may stand for any one of the numbers in \mathbf{N} . Variables can be used to list an infinite number of statements all at once, such as:

$$S(n) : \text{The } n\text{th odd number is } 2n - 1.$$

11. * Check that $S(n)$ is true for $n = 1, 2, 3$.

12. To prove that $S(n)$ is true for every n in the infinite set \mathbf{N} requires checking infinitely many individual statements, an unending amount of work. However, to convince a mathematician it is enough to show how to prove $S(n)$ for the variable n . Since n can stand for any number, the same proof will work in all cases.

Such a procedure is called *induction* and has two steps:

- A. Checking that $S(1)$ is true.
- B. Showing that if $S(n)$ is known to be true, then $S(n + 1)$ must be true.

If both A and B are established, then $S(1)$ is true by A, which means $S(2)$ is true by $S(1)$ and B, so $S(3)$ is true by $S(2)$ and B, so \dots . Note the use of ellipsis.

For example, to prove the statement $S(n)$: “the n th odd number is $2n - 1$,” we check:

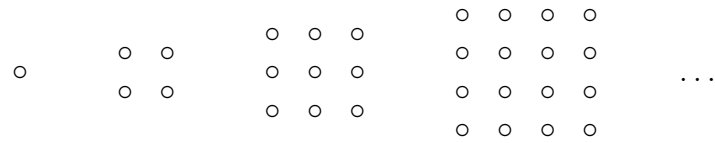
- A: $S(1)$ is true, since for $n = 1$, $2n - 1 = 2 \times 1 - 1 = 1$ gives the first odd number.
- B: Suppose that $S(n)$ is true, namely that $2n - 1$ is the n th odd number. The next, or $(n + 1)$ th odd number is two more, namely

$$(2n - 1) + 2 = 2n + 1 = 2n + 2 - 1 = 2(n + 1) - 1,$$

showing that the statement $S(n + 1)$ is true, so the formula works for the next value $n + 1$ of the variable.

13. * Prove by induction the truth, for all n in \mathbf{N} , of the statement $S(n)$: “the n th even number is $2n$.”

14. Let us return to the sum of the first n odd numbers. Depicting the numbers as dots arranged in an array, we get a sequence of pictures:



Each new odd number adds an extra L-shaped piece consisting of an extra column, an extra row, and a single corner dot. The result is a square array with a square number of dots. We get a *conjecture* that the sum of the first n odd numbers is the square number n^2 . Using ellipsis, the variable n , and the proven formula for the n th odd number $2n - 1$, this may be stated as

$$S(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

15. * State $S(5)$ without ellipsis and check if it is true.
16. * Use induction to prove that the sum of the first n odd numbers is n^2 :
- A: Check $S(1)$.
- B: Suppose that $S(n)$ is known to be true. Add the $(n + 1)$ th odd number to the sum of the first n using the two formulas and some algebra, justifying the steps:

$$\begin{aligned} 1 + 3 + \cdots + (2n - 1) + 2(n + 1) - 1 &= n^2 + 2(n + 1) - 1 \\ &= n^2 + 2n + 1 = (n + 1)^2 \end{aligned}$$

This shows that statement $S(n + 1)$ is true as well.

17. * Using induction, prove $S(n) : \text{the sum of the first } n \text{ even numbers is } n^2 + n$.

18. The sum of the first n odd numbers is an example of a *recursive function*, defined by specifying its first value and then the rule for computing the next value from the previous:

A. $f(1) = 1$

B. $f(n + 1) = f(n) + 2(n + 1) - 1,$

where $2(n + 1) - 1$ is the formula for the next odd number after the n th.

Thus, proving $S(n)$ for all n is the same as solving the recursion for f with the formula $f(n) = n^2$.

19. * There is often more than one way to prove a theorem. Using the formula n^2 for the sum of the first n odd numbers, and the fact that each even number is one more than an odd number, prove that the sum of the first n even numbers is $n^2 + n$.

20. * Prove, however you like, that that the sum of the first n numbers is $(n^2 + n)/2$, also known as $n(n + 1)/2$.

21. * Write the sum of the first n numbers as a recursive function.

22. Induction gives a way to prove rather complicated formulas in little steps. For example, the sum of the first n squared numbers is

$$S(n) : 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$

Admittedly, it is hard to guess this formula, but once guessed it can be checked by induction (with a bit of algebra):

A: Check that $S(1) : 1^2 = 1(1+1)(2 \times 1 + 1)/6$ is true.

B: Assume $S(n)$ and derive $S(n+1)$ by adding $(n+1)^2$ to the formula for n :

$$\begin{aligned} 1^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\ &= \frac{(n+1)[2n^2 + 7n + 6]}{6} = \frac{(n+1)[n+2][2n+3]}{6} \\ &= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}, \end{aligned}$$

which is the formula for $n+1$.

23. * Use induction to prove $S(n) : 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

24. It is sometimes necessary to check more than the first case “by hand.” Using a new variable k for the number of initial cases checked gives a new first induction step.

Furthermore, it is sometimes helpful to use all the previous formulas $S(1), S(2), \dots, S(n)$ to prove that $S(n+1)$ is true. This changes the second step, too, giving the two steps for *strong induction*:

A'. Check that $S(1), \dots, S(k)$ are true.

B'. Show that if $S(1), \dots, S(n)$ are all known to be true, with n at least k , then $S(n+1)$ must be true.

This is equivalent to ordinary induction but with more things spelled out in the proof.

For example, we use strong induction to prove

Theorem 1 *Every number in the Fibonacci sequence is a counting number.*

Proof: Recall that the Fibonacci sequence is generated by the recursion

A1: $f(1) = 1$

A2: $f(2) = 1$

B: $f(n + 2) = f(n + 1) + f(n)$

Now let $S(n)$ be the statement “ $f(n)$ is a counting number.”

$S(1)$ is true since $f(1) = 1$.

$S(2)$ is true since $f(2) = 1$.

Whenever $S(n)$ and $S(n + 1)$ are both true, then $f(n + 2) = f(n + 1) + f(n)$ is the sum of counting numbers, which is a counting number, so $S(n + 2)$ is true. \square

25. No need to worry if some of the initial cases are false, as long as they can be checked without too much trouble. For example, suppose that postage stamps come in 2 cent and 5 cent denominations. We can use strong induction to prove

$S(n)$: n cents postage can be made using 2 cent and 5 cent stamps,

for all $n \geq 4$. First check

- $S(4)$: $4 = 2 + 2$
- $S(5)$: $5 = 5$

Now suppose $n \geq 5$ and we have shown that $S(4), \dots, S(n)$ are true. Since $n + 1 = (n - 1) + 2$, we can form $n + 1$ cents postage by adding a 2 cent stamp to $n - 1$ cents postage, which can be formed from 2 cent and 5 cent stamps since $S(n - 1)$ is true.

Note that $S(1)$ and $S(3)$ are *not* true, but $S(2)$ is true along with $S(4), S(5), S(6), \dots$

26. * Use the same idea to prove that any postage of 8 cents or more can be formed using 3 cent and 5 cent stamps.

27. Strong induction can be used to prove the *least element property* of \mathbf{N} :

Theorem 2 *Every nonempty subset of \mathbf{N} contains a least element.*

This is usually proved as the equivalent *contrapositive* statement:

Theorem 3 *If a subset of \mathbf{N} contains no least element, then it must be empty.*

Proof: Let M be a subset of \mathbf{N} and consider the statement

$$S(n) : n \text{ is not an element of } M.$$

Now suppose that M does not contain a least element. Then

A: $S(1)$ is true, since 1 is the least possible element, so if M contained 1 it would have to be its least element.

B': If $S(1), S(2), \dots, S(n)$ are all true, then all elements of M must be at least $n + 1$. Thus $n + 1$ is not in M , for if it were then it would be the least element. Hence $S(n + 1)$ is true.

We conclude that $S(n)$ is true for every n in \mathbf{N} , which exhausts all the possibilities for elements of M , so M must be empty. \square

28. A *prime number* is a counting number bigger than 1 which is not evenly divisible by any counting number smaller than itself, other than 1. The five smallest prime numbers are 2, 3, 5, 7, 11. Strong induction gives us the prime factorization theorem:

Theorem 4 *Every counting number bigger than 1 is either a prime number or a product of prime numbers.*

Proof: Let $S(n)$ be the statement “ n is either a prime number or a product of prime numbers.” Skip $S(1)$, since it is irrelevant, and check that $S(2)$ is true, since 2 is prime.

Now suppose that $S(2), \dots, S(n)$ are true and consider the number $n + 1$. There are exactly two possibilities:

- $n + 1$ is prime,
- $n + 1$ is not prime.

In the first case, $S(n + 1)$ is true since $n + 1$ itself is prime.

In the second case, $n + 1$ is evenly divisible by a strictly smaller number bigger than one, which we will call a . But then

$$n + 1 = (a) \times ([n + 1]/a),$$

with both factors being counting numbers strictly between 1 and n . But then $S(a)$ and $S([n + 1]/a)$ are both true by the strong induction step B', so $n + 1$ is the product of two numbers each of which is either prime or a product of primes. Conclude that $S(n + 1)$ is true. \square

3 Homework

Consider the recursion

A1. $f(1) = 2$

A2. $f(2) = 2$

B. $f(n + 2) = f(n + 1)f(n)$

- X1.** Compute the values $f(1), \dots, f(6)$ for this recursion
- X2.** Prove that for every n in \mathbf{N} , the value $f(n)$ is a power of 2, namely a number of the form 2^p for some counting number p .
- X3.** Accepting that X2 is true allows us to write $f(n) = 2^{p(n)}$, where $p(n)$ is the counting number power of 2 giving $f(n)$. Write a table of values for $p(1), \dots, p(6)$.
- X4.** Write the recursion for $p(n)$ from X3. Recognize it?
- X5.** Solve the recursion for $p(n)$ in X4, and thus solve the recursion for $f(n)$.

Fix numbers a, b, c and define $f(n) = an^2 + bn + c$ for all n in \mathbf{N} .

- Y1.** Put $a = 2$, $b = 0$, and $c = 1$. Compute the values $f(1), f(2), f(3), f(4)$.
- Y2.** Compute the successive differences $f(2) - f(1)$, $f(3) - f(2)$, and $f(4) - f(3)$ from the function values in Y1.
- Y3.** Find a formula for $f(n + 1) - f(n)$ using the general a, b, c formula for $f(n)$. Simplify it as much as possible.
- Y4.** Use the formula from Y3 to write a recursion for $f(n)$.
- Y5.** Find values for a, b, c such that the recursion for $f(n)$ has the form
- A. $f(1) = 1$.
- B. $f(n + 1) = f(n) + 2n + 1$

Hint: some of the numbers a, b, c could be zero.

4 Homework II

Recall that the Fibonacci numbers $\{f(1), f(2), f(3), \dots\}$ are given by the recursive function

A1. $f(1) = 1$

A2. $f(2) = 1$

B. $f(n+2) = f(n+1) + f(n)$

Z1. Evaluate the first 10 Fibonacci numbers $\{f(1), f(2), f(3), \dots, f(10)\}$.

Z2. Consider the statement

“The n th Fibonacci number is bigger than $2n$.”

Find the smallest Fibonacci number for which it is true.

Z3. Prove, using induction, that the statement in Z2 is true for all $n \geq 10$.

Consider the statement

“Seven to the n th power is one more than three times a counting number.”

W1. Write the statement as a formula using the variable n .

W2. Check the statement for the first two powers of 7, namely $7^1 = 7$ and $7^2 = 7 \times 7$.

W3. Prove, using induction, that the statement is true for all $n \in \mathbb{N}$.