Hilbert's Wonderful Infinite Hotel! September 23, 2012

Kyle Sykes
Washington University Math Circle

(This presentation was inspired by the book *The Cat in Numberland* by Ivar Ekeland, which I highly recommend checking out from your local library.)

1 The Hotel

You have just purchased the Hotel Infinity! The hotel is a wonderful place, but it might be a bit confusing at first so let me show you around!

First, the hotel has many, many rooms. In fact, it might even be the biggest hotel ever built! Every room is numbered in increasing order by the numbers $1, 2, 3, 4, 5, \ldots$. This means there is a room 25, a room 100, and a room 198,491,329.

Problem 1. *How many rooms does your hotel have?*

Next, let's put some guests into these rooms. Let's say that Number 1 is staying in room 1, Number 2 is staying in room 2, and Number 198,491,329 is staying in room 198,491,329, and so on.

Problem 2. *Is the hotel full?*

What a wonderful hotel! I know you will do just fine managing it. You will find that the guests are very willing to work with you on any issues that come up. Above all else, *you must make sure the hotel stays full at all times!* We have a business to run and we can't go around leaving rooms empty, right?

2 A New Guest Arrives

One dark and terribly stormy night, a guest comes into the Hotel Infinity looking very wet, and very miserable. He introduces himself to you saying, "Hello, my name is Number 0, and I would very much appreciate if you had a room I could stay in. Do you have a room available?".

The other guests all agree that you shouldn't send Number 0 back out into the stormy night, but they are not willing to share a room because a bunch of issues come up (for instance, if you put Number 0 in with Number 1, you can't tell them apart from Number 10 and that would make things very confusing!). All the numbers turn to you and ask you what they should do to make room for Number 0.

Things to keep in mind:

1. The hotel needs to stay full.
2. You can't put two numbers into the same room, otherwise everything gets confusing.
3. The other guests are very willing to cooperate with your instructions, but they would rather not be forced to leave the hotel unless they want to (which they do not right now!)

Problem 3. *Find a way to fit Number 0 into the hotel! Explain your process, as well as telling me how to find any guest that I want. For instance, if I go looking for Number 7 I'd like to know what room he is staying in.*

Problem 4. *Was the hotel full before Number 0 came? If so, how is it that there is room for Number 0? What happened?*

3 The Alphabet

So, we managed to find room for Number 0, and everyone is happy with their new arrangements. Number 0 gets along with every other number so much that he decides to stay for a while.

Number 0 tells the other numbers of a group of friends of his called the Alphabet. They are 26 letters who he met while traveling the world, and they know how to have a good time! He offers to invite the letters over to the hotel, and all the numbers agree because they love making new friends.

So the alphabet comes to town and Letters $A, B, C, D, \ldots, X, Y, Z$ walk into the hotel lobby. Everyone is so excited, and they all play and have fun until night falls on the hotel. It is dark outside now and the letters *really* want to stay the night and get to know all the numbers better, but the hotel is still full and there are sadly no rooms available. They all turn to you and say, "Please, you were able to find room for Number 0, surely you can find room for 26 more, right?" You begin to ponder, wondering how in the world you are going to fit 26 more guests into a hotel that is already packed full!

Problem 5. *Find a way to fit the 26 letters of the alphabet into the hotel!*

4 Going to Visit New Friends

As the letters all prepare to leave the hotel the next morning they say, "You numbers should come stay at our house sometime! We will have a grand old time!" The numbers are all very excited, because they really enjoyed their new friends. They all decide that they want to go visit the letters. As they all pack up and get ready to go, you stop them. "You can't all leave the hotel at once! I have bills to pay, so maybe half of you could go now, and half of you could go later". The numbers all agree this is the wisest choice because they love the hotel and want to see it stay open. They all turn to you and ask you how to send half of the numbers to visit the letters.

Problem 6. *Find a way to split the numbers into two groups so exactly half of them stay in the hotel, and half of them leave.*

Now the first half of the numbers say "Farewell!" and go off to visit the letters for one week.

Problem 7. *How full is the hotel? Half-full, half-empty, or something else entirely..?*

After a couple of nights, the numbers grow restless and can't sleep. They come to you and say, "We cannot continue to sleep this way! We have empty rooms on either side of us, and so we feel alone and cannot sleep! You must find a way to rearrange us so that we have someone on both sides of our room, so that way we don't feel so alone!"

Problem 8. *Find a way to arrange the remaining numbers so they are all next to each other and therefore happy.*

Problem 9. *Is the hotel full? Why or why not? How does this compare to your answer above? What happened??*

After a week, the first group of numbers comes back from the letters, and the second group goes to visit the letters and in the end everyone is very happy and relaxed and everything goes back to normal.

5 The Negative Numbers

The alphabet says their goodbyes, and all the Numbers go back to the way things were before the alphabet stayed the night. Everyone is happy, although the letters spread news about your amazing hotel. They say “It’s a hotel that is always full, yet they somehow find room for everyone that wants to stay!” It’s quite a marvel, and so naturally word gets out.

One day, a large group of new numbers appear. They call themselves The Negative Numbers, and they resemble the Numbers currently staying in the hotel quite a bit. Number -1 looks similar to Number 1, Number -7 looks similar to Number 7, Number -198,491,329 looks like Number 198,491,329, and so on. The numbers are very excited about the arrival of the negative numbers are wish for them to spend the night to make even more new friends. The negative numbers are excited about staying in your famous Hotel Infinity. They all turn to you and say “Can you help us all find a room?”

Problem 10. *Can you find a way to fit all the numbers and the negative numbers into the hotel?*

6 The Rational Numbers (ie: Fractions!)

The negative numbers are amazed at the Hotel Infinity, and they say goodbye and promise to spread the word of your amazing hotel. Things are back as usual in the hotel, when a new batch of guests arrives, the Rational Numbers.

These are numbers that come from a far off land, yet look similar to your numbers here. They are a bit odd though, as one number carries another on it's head! Some examples of rational numbers are $\frac{1}{2}$ and $\frac{134}{157}$. How very odd! The numbers in the hotel are naturally very excited by the new guests, and they really want them to stay in the hotel for a bit. They all turn to you and say "Can you find everyone a room in your hotel?"

Problem 11. *Can you find a way to fit all the numbers into the hotel? What difficulties do you encounter? Do you think it is possible?*

You think for a while, then decide it would be best if the numbers all lined up in rows for better organization. You tell them to arrange themselves in the following way:

- All the numbers $1, 2, 3, 4, \ldots$ line up in the first row in increasing order.
- In the second row, the fractions with 2 in the denominator(that is, the bottom) line up behind the numbers in the first row so that $\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \ldots$
- In the third row, the fractions with 3 in the denominator line up in the third row so that $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots$
- and so on...

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Now that you have all the numbers lined up in these neat rows so the task should be easy, right?...*right??*

Problem 12. *Find a way to fit all the numbers into the hotel? How did you do it?!*

7 Homework for Next Time!

These relate to what we did this week and to what you did the last few weeks with inductio and recursion.

Problem 13. *After some unfortunate events regarding a soda party in room 8,675,309, you have decided that bottles of soda are no longer allowed inside the hotel, and no soda may be brought into the hotel. Despite this, the guest in room 1 goes to the guest in room 2 to get a soda. The guest in room 2 goes to room 3 to get two sodas–one for himself and one for the guest in room 1. In general, the guest in room* N *goes to room* $(N+1)$ *to get* N *sodas. They each return, drink one soda and give the rest to the guest from room* $(N-1)$*. Thus despite the fact no sodas have been brought into the hotel, each guest can drink a bottle of soda inside the hotel. What is going on here? I thought you said no soda?!*

Problem 14. *In your hotel there are infinitely many janitors to attend to the rooms. The*

first janitor goes along the corridor turning ***on*** *the light of every room. The second janitor goes down the corridor and turns the lights* ***off*** *in every second room. The third janitor goes down the corridor flicking the light switch of every third room (either on or off). And so it continues, with the* n^{th} *janitor flicking the light switch of every* n^{th} *room. Which lights are left on in the end, and why?*

Problem 15. *In the solution to problems in Section 6, you (likely) made an argument about how to arrange the rational numbers in such a way to easily count them. This had you count the rational number* $\frac{1}{2}$*, and also count the numbers* $\frac{2}{4}, \frac{3}{6}, \frac{4}{8} \ldots$

First, tell me what is the relationship between these numbers. Was there something special in choosing $\frac{1}{2}$*? What if I had considered* $1, \frac{2}{2}, \frac{3}{3}, \ldots$*? How many times did we count each number when we solved the problems in Section 6?*

Second, can you find a way to list every rational number (fractions) so that no rational number appears twice in the list?