HOTEL INFINITY: MANY, MANY, MANY, ... MORE GUESTS

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Last week you stayed at Hotel Infinity (Hotel ∞), and what a wonderful place it was! The single tower of Hotel ∞ rises up into the sky as far as your eyes can see. Each floor has a single room and there is a room for every natural number 1, 2, 3, ... Just imagine riding the elevator to room 198,491,329.

1. Exchanging keys

You may remember that one evening last week a new guest arrived, just as every room in Hotel ∞ was occupied. Luckily, the guests staying at Hotel ∞ are willing to exchange their keys for new ones to make a room available. Each guest has a key with a natural number printed on it. The manager wants to exchange old keys for new one in the following way.

(1) If the old key is labeled with an odd number, then the guest gets a new key with twice that number.
(2) If the old key is labeled with an even number, then the guest gets a new key with that number plus 2.

Problem 1. Give some examples of how keys are exchanged. What do you think of the manager's plan?

Problem 2. Consider a second exchange rule: If the old key is labeled with the natural number n, then the guest gets a new key with number 2^{n-2}. Give some examples of how keys are exchanged. What do you think of this exchange rule?

Problem 3. What would be a valid way to exchange keys? Represent your way to exchange keys with arrows in Figure 1. Write down a rule that tells the manager how to exchange keys.

Problem 4. Consider a third exchange rule:

(1) If the old key is labeled with an odd number, then the guest gets a new key with twice that number.
(2) If the old key is labeled with an even number, then the guest gets a new key with that number plus 1.

Is this a valid way to exchange keys?

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We can think of a rule to exchange keys as a function f. If the old key is labeled n, then the new key is labeled $f(n)$. A *valid* exchange rule must satisfy the following two properties:

(1) For every natural number n, the function value $f(n)$ is a natural number.
(2) For any two natural numbers n and m, if $n \neq m$, then $f(n) \neq f(m)$.

The first property guarantees that each guest gets a new room, while the second property makes sure that two guests from different rooms do not end up with the same new room. Mathematically, we say that (1) the image $f[\mathbb{N}] \subseteq \mathbb{N}$ and (2) the function f is *injective*.

2. Hotel Even and Hotel Prime

There are two other hotels in the historic city center: Hotel Even and Hotel Prime. Build before Hotel ∞, these hotels do not seem to have as many rooms. Hotel Even only has rooms with even rooms, while Hotel Prime uses only prime numbers as room numbers. You can see these two hotels in Figure 2. Although Hotel ∞ is full at the moment, Hotel Even is empty. The manager of Hotel ∞ wonders whether it is possible to house all guests of Hotel ∞ in Hotel Even.

Problem 5. Use Figure 2 to represent a way to give a key of Hotel Even to each guest in Hotel ∞. Write down a valid rule to exchange keys.

Problem 6. Similarly, find a way to house all guests of Hotel ∞ in Hotel Prime.

Hotel ∞, Hotel Even, and Hotel Prime correspond to three sets: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathsf{E} = \{2, 4, 6, \dots\}$, and $\mathsf{P} = \{2, 3, 5, \dots\}$. The set of even numbers E is a subset of the set of all natural numbers $\mathbb{N}$, which we write as $\mathsf{E} \subseteq \mathbb{N}$. In fact, E is a *proper* subset of $\mathbb{N}$, because some natural numbers are in $\mathbb{N}$ but not in E (which ones?). This is sometimes written as $\mathsf{E} \subsetneq \mathbb{N}$. Although $\mathbb{N}$ properly contains E, as a set $\mathbb{N}$ is no larger than $\mathbb{E}$ because there is an injective function $f\colon \mathbb{N} \to \mathsf{E}$.

3. Hotel $\infty \times 2$

At the lake, construction on a new hotel is in full swing — Hotel $\infty \times 2$. Just like Hotel ∞, this hotel will have infinitely many rooms. But rather than one tower, Hotel $\infty \times 2$ will have two towers to house guests. Instead of a key with a single natural number, guests staying in Tower 1 will have room numbers like $(1, 1729)$. Of course, there is also a room 1729 in Tower 2, which will have room number $(2, 1729)$.

Problem 7. The manager of Hotel $\infty \times 2$ is very excited that the new hotel will have twice as many rooms as Hotel ∞. What do you think?

Problem 8. Use Figure 3 to show that even if Hotel $\infty \times 2$ is completely full, all guests can get a room in Hotel ∞. Can you write down a rule to exchange keys?

Problem 9. Somewhat disappointed that Hotel $\infty \times 2$ is no larger than Hotel ∞, the manager wants to build an additional third tower. Do you think Hotel $\infty \times 3$ is any larger? Use Figure 3 to explain your reasoning.

Of course, you can now image larger and larger hotels. Hotel $\infty \times 2$ has two towers, Hotel $\infty \times 3$ three, $\ldots$, and Hotel $\infty \times 4,294,967,297$ has a staggering number of towers, each with infinitely many rooms. However, none of these hotels has any more rooms than Hotel ∞, as you will now demonstrate to the managers of Hilbert's Hotels.

Consider an arbitrary natural number M. Hotel $\infty \times M$ has M towers. Each tower has a room for every natural number n. Each guest is given a key labeled with a pair of natural numbers (m, n). The first number m tells a guest in which tower they are staying. The second number n directs the guest to the right floor within that tower.

Problem 10. Show that Hotel ∞M has no more rooms than Hotel ∞ by finding a rule that exchanges keys. *Hint:* Consider the function $f(m, n) = 2^m \cdot 3^n$ and check that this is a valid rule to exchange keys.

Problem 11. What about a Hotel ∞^2, which has infinitely many towers, each with infinitely many rooms?

4. Hotel ∞^3

Finally, the managers of Hilbert's Hotels realize that building more towers (even with infinitely many rooms) will not make a hotel have any more rooms. Because they are still a bit nervous that they might have to turn down guests, they propose to construct one more hotel, Hotel ∞^3. Of course, Hotel ∞^3 will have infinitely many towers, and each tower will have infinitely floors, but now each floor will have infinitely many rooms, rather than just one. To make sure that guests can find their way around the hotel, each key will have three natural numbers (a triple). For example, the key labeled $(3, 4, 5)$ will open the fifth room on the fourth floor in the third tower.

Problem 12. Convice the managers that even if Hotel ∞^3 is fully occupied, you can move all their guests into Hotel ∞.

5. Hotel ∞^∞

Just as you have convinced the managers of Hilbert's Hotel that no infinite hotel is larger than another, you get a phone call from the manager of Hotel ∞^∞. Hotel ∞^∞ is currently full, and the manager would like to know if you could fit all guests into Hotel ∞. As you talk to the manager on the phone, you start to realize that Hotel ∞^∞ is a rather strange place. Each guest has a key with infinitely many numbers! Here are three examples of possible room numbers:

(1) $(1, 1, 1, 1, \dots)$
(2) $(1, 2, 3, 4, \dots)$
(3) $(2, 4, 6, 8, \dots)$

Problem 13. Use the following table to assign rooms in Hotel ∞ to these guests staying in Hotel ∞^∞.

Room in Hotel ∞	Guest from Hotel ∞^∞
1	
2	
3	
4	
⋮	

Problem 14. Is there a guest from Hotel ∞^∞ you have not given a room in Hotel ∞ yet?

Without warning, infinitely many guests from Hotel ∞^∞ arrive. You assign them rooms in Hotel ∞ as best as you can. You have a feeling, however, that there must be some guests from Hotel ∞^∞ you have not given a room in Hotel ∞.

Problem 15. Are you correct? Why (not)?

Problem 16. What does this tell you about the relative sizes of Hotel ∞ and Hotel ∞^∞?

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Figure 1: A new guest arrives at Hotel ∞.

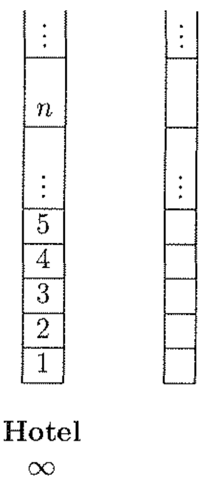

Figure 2: Hotel ∞, Hotel Even, and Hotel Prime.

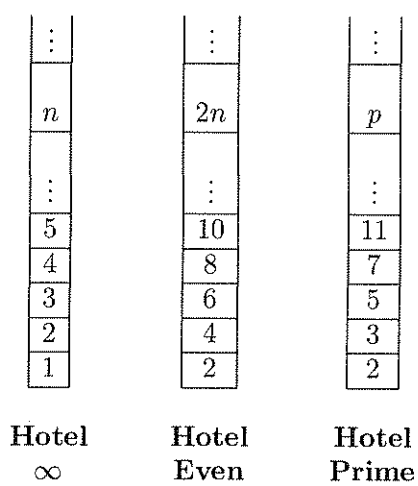

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Figure 3: Comparing Hotel ∞, Hotel $\infty \times 2$, and Hotel $\infty \times 3$

⋮
n
⋮
5
4
3
2
1

Hotel ∞

⋮	⋮
(1, n)	(2, n)
⋮	⋮
(1, 5)	(2, 5)
(1, 4)	(2, 4)
(1, 3)	(2, 3)
(1, 2)	(2, 2)
(1, 1)	(2, 1)

Hotel $\infty \times 2$

⋮	⋮	⋮
(1, n)	(2, n)	(3, n)
⋮	⋮	⋮
(1, 5)	(2, 5)	(3, 5)
(1, 4)	(2, 4)	(3, 4)
(1, 3)	(2, 3)	(3, 3)
(1, 2)	(2, 2)	(3, 2)
(1, 1)	(2, 1)	(3, 1)

Hotel $\infty \times 3$

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A function f is *injective* if for all m and n, if $m \neq n$, then $f(m) \neq f(n)$.

Problem 1.

(1) Can there be an injective function from the set $\{1, 2, 3, 4\}$ to the set $\{1, 4, 9\}$? Why (not)?
(2) Can there be an injective function from the set $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ to the set $\mathbb{N} = \{1, 2, 3, \dots\}$? Why (not)?
(3) Can there be an injective function from the set $\mathbb{N}$ to the set $\mathsf{E} = \{2, 4, 6, \dots\}$? Why (not)?
(4) Find an injective function from the set $\mathbb{N}_0$ to E.

If it is possible to find an injective function from X to Y, then we write $|X| \leq |Y|$. We say that the *cardinality* of X is less than or equal to the cardinality of Y.

Problem 2. Use this notation for $\mathbb{N}_0, \mathbb{N}$ and $\mathbb{E}$. Is $|\mathbb{N}_0| \leq |\mathbb{E}|$? Is $|\mathbb{N}_0| \leq |\mathbb{E}|$? Explain your answers.

Problem 3. Let $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots, \}$. Is $|\mathbb{Z}| \leq |\mathbb{N}|$? Why (not)?

Problem 4. The set $\mathbb{N} \times \mathbb{N}$ is defined as follows:

$$\mathbb{N} \times \mathbb{N} = \{(m, n) \colon m \in \mathbb{N}, n \in \mathbb{N}\}.$$

Is $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$?

Problem 5. The set $\mathbb{N}^{\mathbb{N}}$ is defined as follows:

$$\mathbb{N}^{\mathbb{N}} = \{(n_1, n_2, n_3, \dots) \colon n_i \in \mathbb{N} \text{ for every } i \in \mathbb{N}\}.$$

Is $\left|\mathbb{N}^{\mathbb{N}}\right| \leq |\mathbb{N}|$? Explain your reasoning.

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