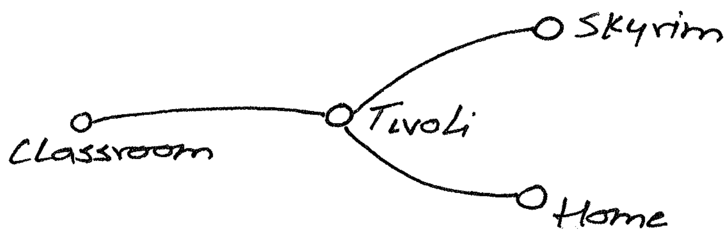
GRAPH THEORY

Consider the following problem:

It's the middle of winter and it's cold and snowing heavily outside. Inside the classroom however, it's warm and cosy. A chalk duster sits on the table, trying to look as ordinary as can be. The duster is, however a portkey connecting the class to the Tivoli movie theater. At the theater, between the screens 9 and 10 is a dustbin and a little popcorn machine both of which are also portkeys. The popcorn machine takes you to the world of Skyrim while the dustbin takes you home. What is the route from the classroom to your home, using portkeys to avoid getting caught in the snow?

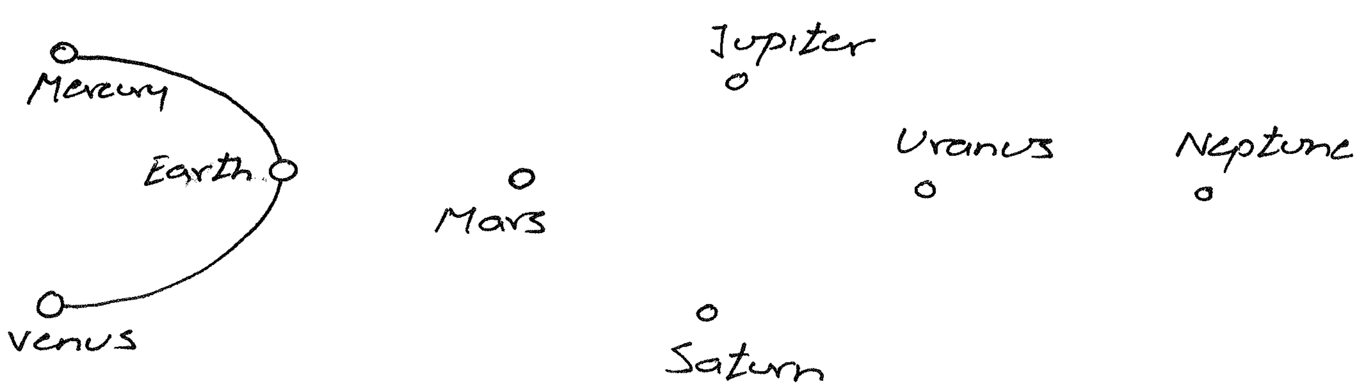
The answer is easy: Use the duster to go from the classroom to the theater and the dustbin to take you home. Which of-course makes you wonder, what was the point of the rest of the details? Many real-world problems start out this way, with a mess of unnecessary details that hide the real relations between the different quantities. One way that mathematicians get rid of all this extraneous data is by drawing graphs.

A graph is a diagram with points (called *vertices)* and connecting segments (called *edges*). For example the graph of the above problem is the following:



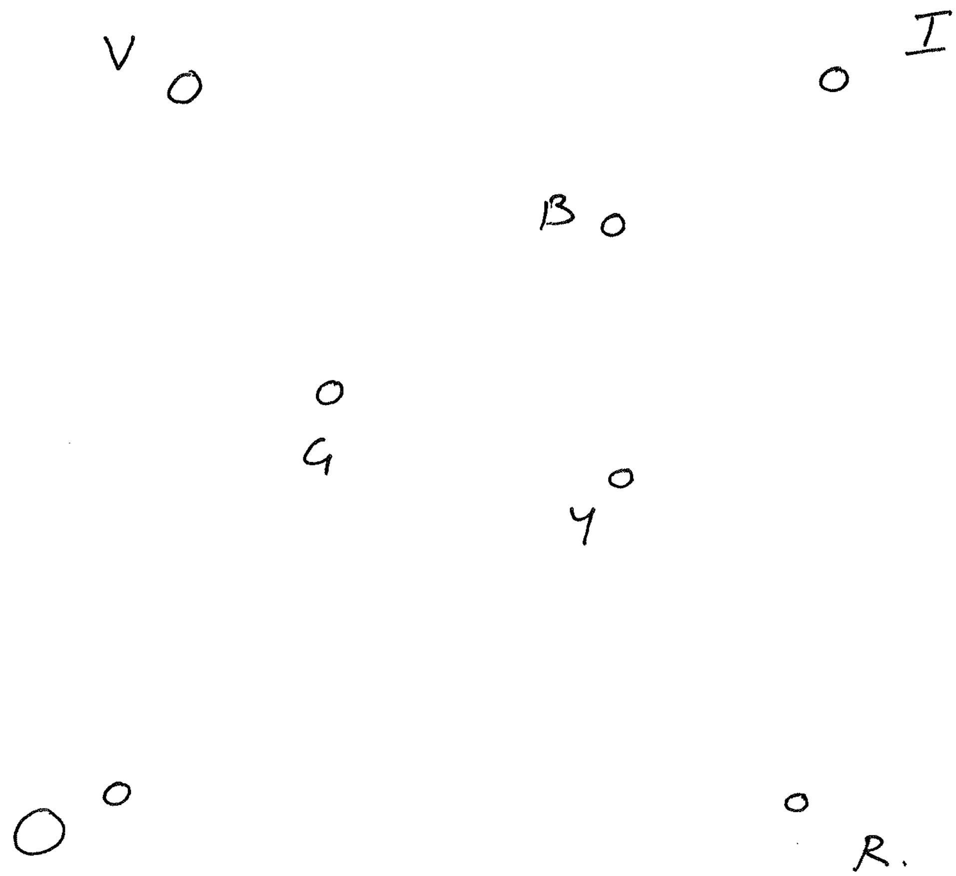
Let us try to draw graphs for some other problems:

Q1. In the future when men live on every planet of the solar-system there is a plan to build a communications network that connects all the planets. The plan introduced by AT&T involves building a broadband connection from earth to mars, venus and mercury, from mars to jupiter and saturn and from uranus to jupiter, saturn and neptune. Then join the vertices drawn below by edges in a way that captures this data:



Now, here's a tricky one. Draw a graph for the details given below:

Q2. The seven colours Violet, Indigo, Blue, Green, Yellow, Orange and Red have an argument about where to live in the clouds when not called on to stand in a line by the Rainbow. Violet is a good friend of Indigo and says she'll stay next to her. Blue, Green and Yellow are in the same yoga class and want to stay next to each other. Orange has a crush for Red and wants to live next to her while Red wants to live next to both Orange and Indigo. Violet, green and orange go to the same football practice and would like to live next to each other. If each colour is a vertex of a graph, and two colours living next to each other is represented by an edge, draw a graph that displays all the relations that were mentioned above:



Q3. The ten digits 0, 1,..., 9 are friends on Facebook. Checking their friends-list, it was observed that all the even digits were friends with each-other while each odd digit had only the even digits immediately before and after them on their friends-list. Also, 9 was a friend of 0. Draw a graph that shows their friends-network on Facebook.

Show your answers for this section to a mentor and get the next section.

Section I: Total Degree of a Graph

Okay... now that you know how to convert real-world problems into a graphs let us try some calculations.

Q. Looking at the three graphs we drew above, answer the following questions:
a) How many neighbors does Green have?

b) How many neighbors does Orange have?

c) How many planets can you talk to from Mars directly, without having to go through any of the other planets?

d) How many planets an you talk to from Jupiter directly, without having to go through any of the other planets?

e) How many Facebook-friends does 0 have?

f) How many Facebook-friends does 1 have?

Write your answers before turning the page.

Answers:
a) 4

b) 3

c) 3

d) 2

e) 6

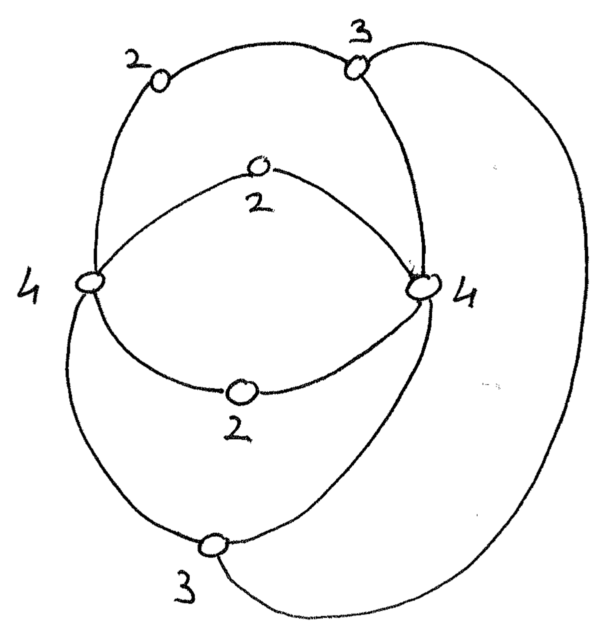
f) 2

In each case the answer was obtained by counting of the number of edges that met at the vertex you were interested in. This is called the <u>degree</u> of the vertex.

The <u>Total degree of a graph</u> is the sum of all degrees of all vertices of the graph.

Calculate the Total degree by adding up the degree of all the vertices for the following graphs:

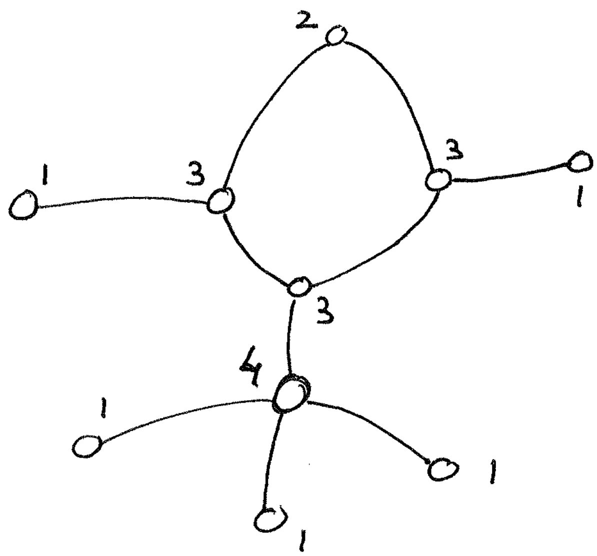
1.



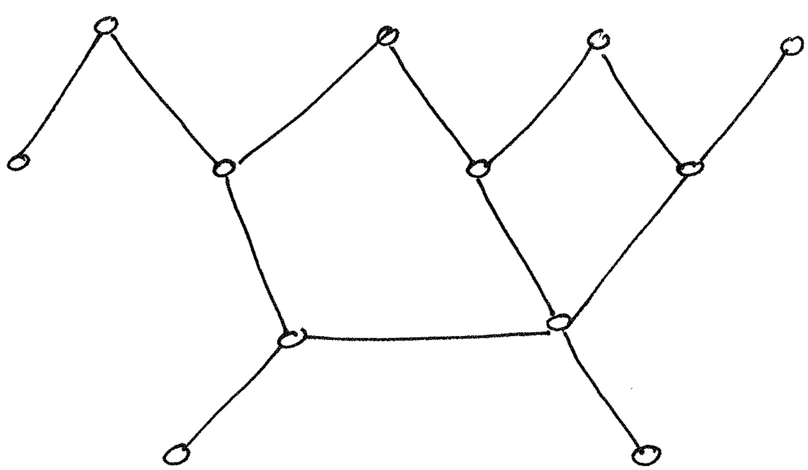
Total Degree:
$2+3+4+2+4+2+3$

$=$

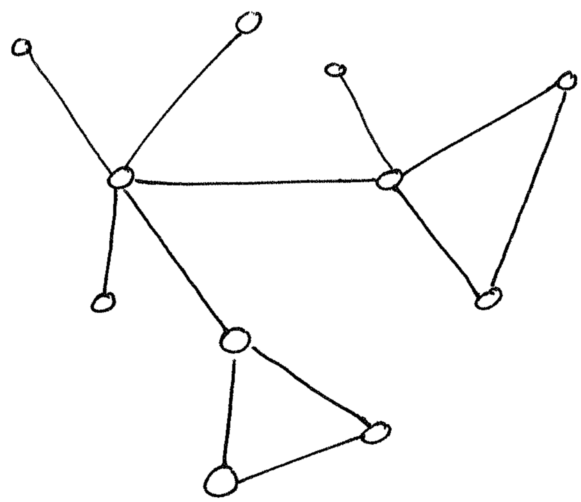
2. Calculate the Total Degree of the following graph:



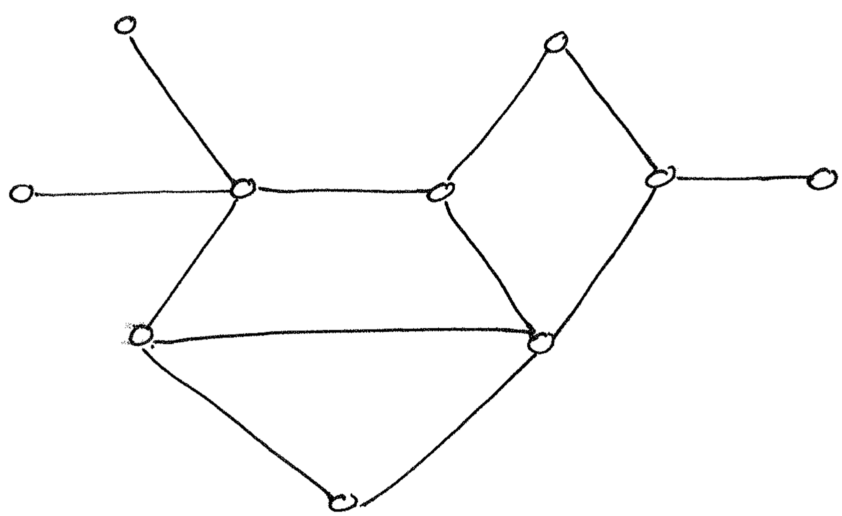
3. Calculate the Total Degree of the following graph:



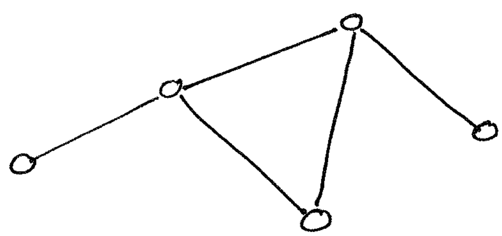
4. Calculate the Total Degree of the following graph:



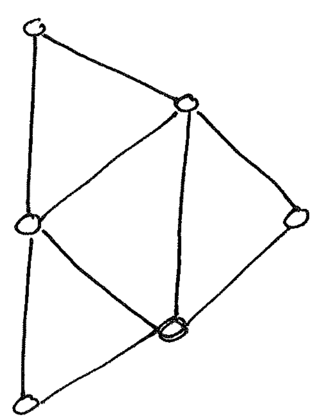
5. Calculate the Total Degree of the following graph:



6. Calculate the Total Degree of the following graph:



7. Calculate the Total Degree of the following graph:



Show your answers to a mentor for the next section.

Section III: Odd Degree vertices of a graph

Observe that in all the 7 graphs of the previous section, the total degree is always even. Why do you think this is so? Discuss with your partner and write down a reason why the total degree of every graph is always even.

Show your answer to a mentor for the rest of this section.

Section III (continued)

Consider the following problem:

Q1. The Mad Hatter invited 4 friends to his tea-party. The hatter is known far-and-wide for his excellent bacon-flavored tea. So they all rush to his little mushroom-house in the glade and arrive at the same time. They find him wearing a bright pink suit and hopping in a circle singing:

> To those of you I will serve tea
> to those of you who swear to me
> that you shook the hands with only three.

Now all 4 of his friends want to drink bacon-tea, but they'll be invited in only if each person shakes hands with exactly three of the others. How should they go about shaking hands so that they all get invited? Draw a graph with 4 vertices for the 4 friends and join any two friends by an edge only if the two of them shake hands.

Q2. Suppose the Hatter had invited 5 friends instead. How should they shake hands so that each guest shakes hands with exactly three others? Can you draw such a graph?

Q3. Suppose the Hatter had invited 6 friends instead. Can you draw a graph where each friends shakes hands with three others (i.e, degree of each vertex is 3) ?

Q4. Suppose the Hatter had invited 7 friends instead. Can you draw a graph where each friends shakes hands with three others (i.e, degree of each vertex is 3) ?

Q5 Suppose the Hatter had invited 8 friends instead. Can you draw a graph where each friends shakes hands with three others (i.e, degree of each vertex is 3) ?

You might have observed a pattern here... If there were an odd number of guests at the party, no matter how they shook hands, all of them would never get bacon-tea.

In terms of their graphs, this means that there does not exist any graph with odd many vertices when each vertex is of degree 3 ! Go back to what you learned about the total degrees in the last section and try to guess why this might be the case. Discuss with your partner and write the reason below.

Show your answers to a mentor for the next section.

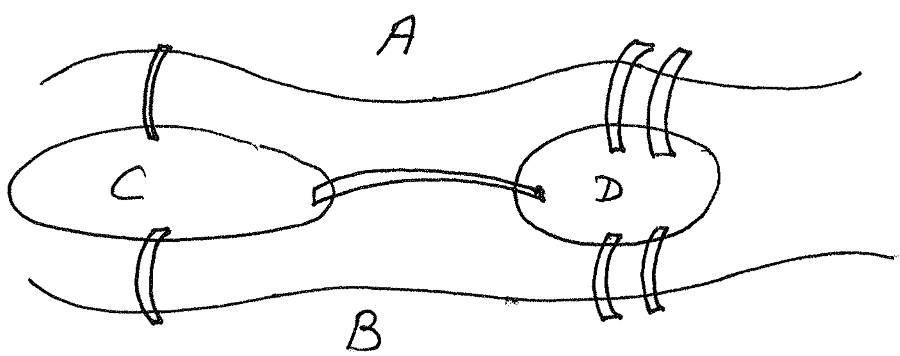
In fact, because of the following observation:

(Odd number) + (Odd number) = (Even number)
(Odd number) + (Even number) = (Odd number)
(Even number) + (Even number) = (Even number)

Using exactly the same argument you wrote just now, it is easy to conclude that in any graph you can only have evenly many vertices which have odd degree (such as degree 3).

Section IV: Eulerian Trails

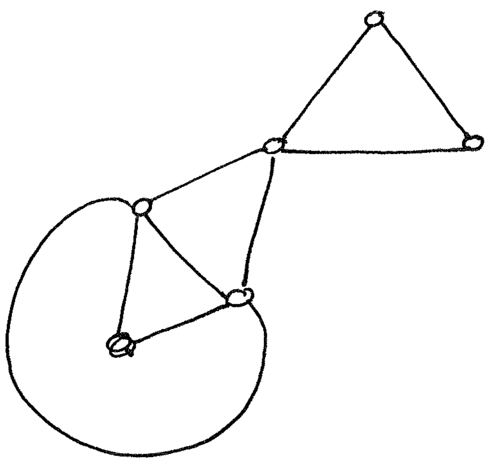
Now we come to the famous Konigsberg problem, which gave birth to Graph Theory. In the eighteenth century, the city of Konigsberg consisted of islands where two branches of the river Pregel joined. (today the city is called Kalingrad, and is in Russia, on the Baltic sea.) Seven bridges connected various islands as shown below.



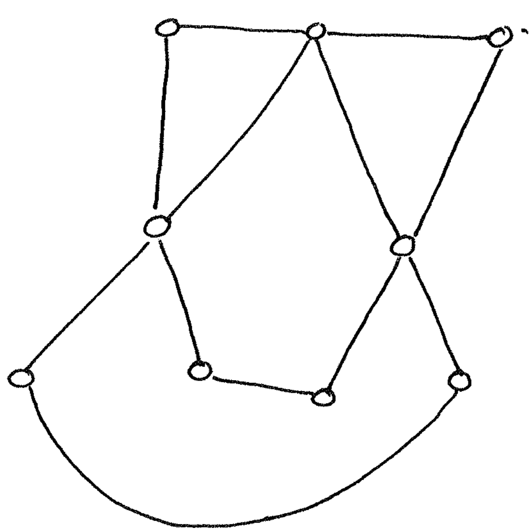
In 1736, the most prolific mathematician of all times, Leonhard Euler, became interested in the following question. Is it possible to walk through town, starting and ending at the same place, so that we walk each bridge exactly once? What do you think? Draw a graph for this problem and try to trace out such a walk.

The following graphs correspond to bridges and islands in different cities around the world. Can you walk across each bridge exactly once and come back to your starting point?

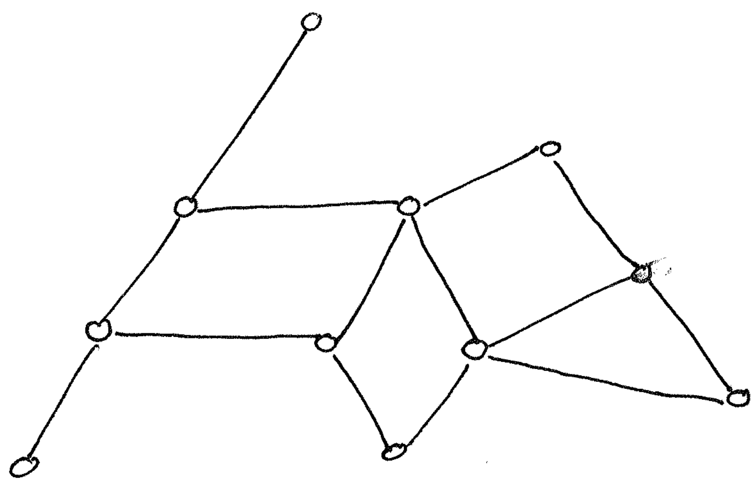
Q1. Try to trace out a closed trail that starts and ends at the same vertex and passes through each edge exactly once. Such a trail may not exist.



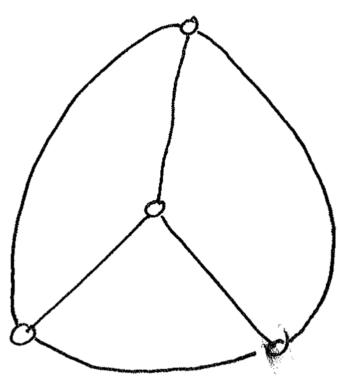
Q2. Try to trace out a closed trail that starts and ends at the same vertex and passes through each edge exactly once. Such a trail may not exist.



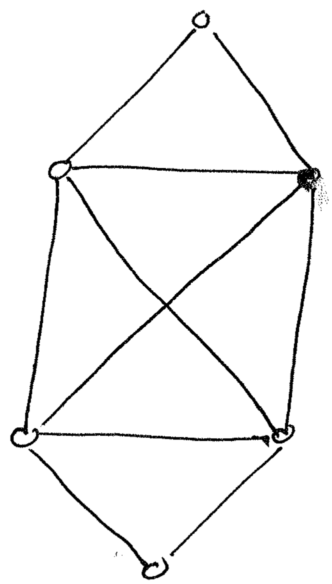
Q3. Try to trace out a closed trail that starts and ends at the same vertex and passes through each edge exactly once. Such a trail may not exist.



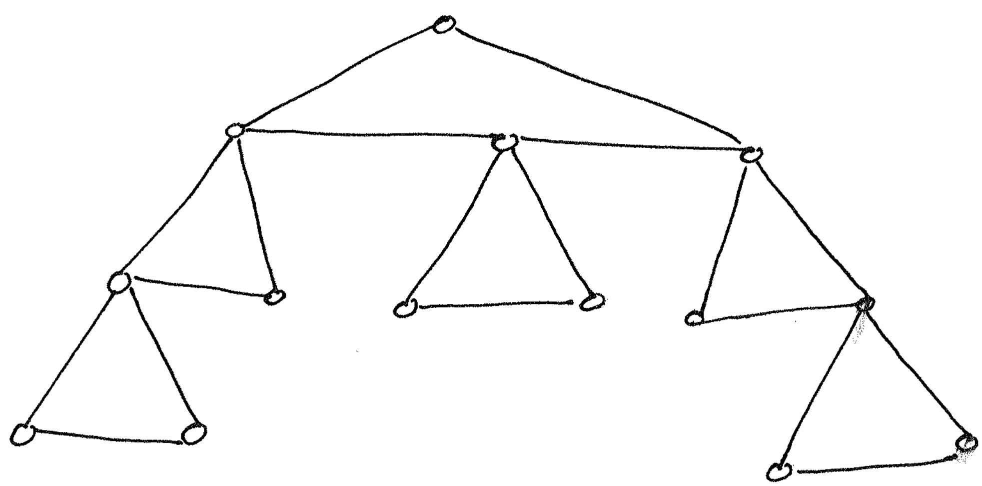
Q4. Try to trace out a closed trail that starts and ends at the same vertex and passes through each edge exactly once. Such a trail may not exist.



Q5. Try to trace out a closed trail that starts and ends at the same vertex and passes through each edge exactly once. Such a trail may not exist.



Q6. Try to trace out a closed trail that starts and ends at the same vertex and passes through each edge exactly once. Such a trail may not exist.



Show your answers to a mentor for the rest of this section:

Such a closed trail is called a *closed Eulerian trail* (after the mathematician Leonhard Euler). Now in all six of the above problems list the degrees of each of the vertices in the table below:

Graph	Degree of each vertex	Does it have an Eulerian trail?
1	2, 2, 4, 4, 4, 2	Yes
2	2, 4, 2, 4, 4, 2, 2, 2, 2	
3		
4		
5		
6		

What is the relation between the degrees of the vertices and whether the graph has a closed Eulerian trail or not? If a graph has a closed Eulerian trail then the degree of each vertex must be even. Why do you think this is so? Discuss with your partner and write your answer below:

Write your answers and show them to a mentor for the next section.

In fact, this is true both ways... a connected graph with a closed Eulerian trail has each vertex of even degree and if a connected graph has every vertex of even degree then the graph does in fact have an Eulerian trail. Let us try to use this fact.

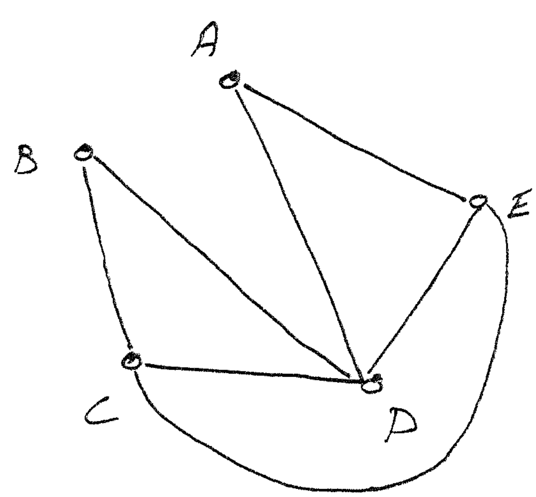
Q. Suppose an island nation consists of 19 islands A, B, C, D, E and F. There are either 2, 4 or 6 bridges that lead into each island. If you can drive from any one island to another via some sequence of bridges then can you drive over each bridge exactly once and come back to your starting point? What if island A had 3 bridges instead? Write your answer below.

You see now, without our graph theory background this would seem to be an impossible problem because we don't know which two islands are connected by bridges. And yet now that we have Euler's result with us we can say that because the degree of each vertex in the graph is even there is a closed Eulerian trail which is the route we would drive along.

Section V: Hamiltonian Cycle

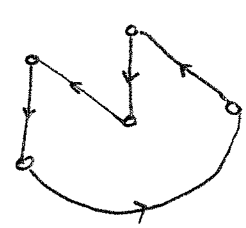
Now, let's try the following problem. You invite n friends over for a party. You want to seat everyone around a circular table. Is it possible to find a seating arrangement so that each guest is a friend of both people seated next to him/her? When the number of people n is large, this becomes a surprisingly difficult problem to compute. Lets try it out:

Q1 Suppose Al, Bob, Chump, Dodo and Ed are invited to your party. Al is a friend of Ed, Chump is a friend of both Bob and Ed while Dodo is a friend of everyone. The graph below show their friend-network. To find a seating arrangement we need a cycle (a closed loop) that goes through every vertex of the graph as shown below:



Then the arrangement we need is:

A—D—B—C—E



Q2. Now you try it. Suppose you invited over friends called A, B, C, D, E and F. C is a friend of A, B, D and E. D is a friend of B, C, E and F. F is a friend of B, D and E. Draw a graph with this friend-network and find a seating arrangement in a circle so that each guest is a friend of both his neighbors.

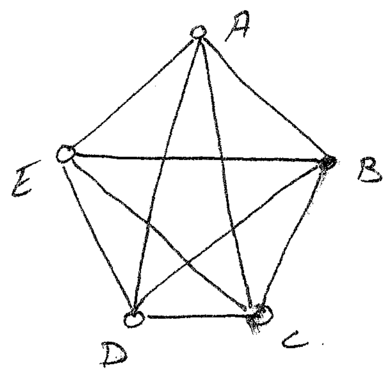
Q3. Suppose you invite over the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. Suppose two numbers know each other only if one of them divides the other. Can you seat them in a circle as required?

In each of these problems, we are attempting to find a closed trail that goes through each vertex exactly once. Such a closed trail is called a *Hamiltonian cycle* (after the mathematician William Hamilton).

We could get a neat way of determining when a graph has a closed Eulerian trail in section IV. Unfortunately, there is no such simple recipe for determining when a graph has an Hamiltonian cycle, but if a graph satisfies certain conditions then it always has a Hamiltonian cycle.

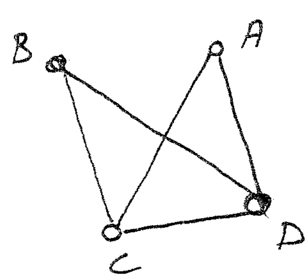
For example, if each of your friends knew every other friend you invited over then clearly any seating arrangement would do the job. Such a graph is called a complete graph and denoted by K_n.

Q Draw the complete graph K_5, ie, a graph with 5 vertices where each vertex is joined to every other.



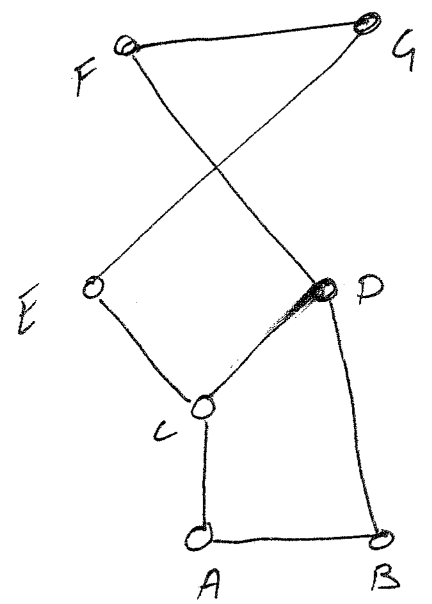
Hamiltonian Cyle :

Q1. Find a hamiltonian cycle in the following graph, if there is one:

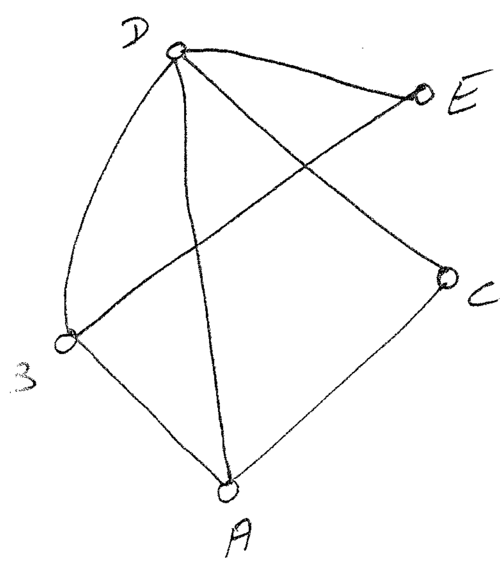


Hamiltonian Cycle :

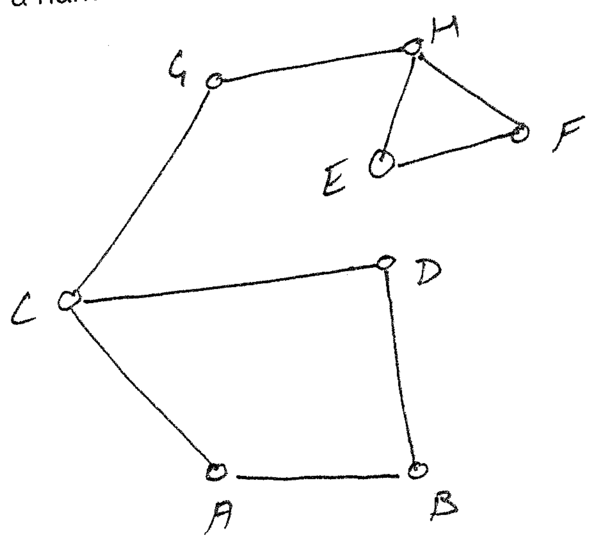
Q2. Find a hamiltonian cycle in the following graph, if there is one:



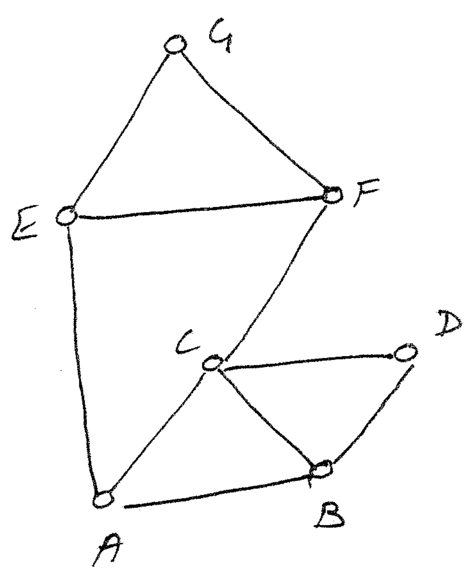
Q3. Find a hamiltonian cycle in the following graph, if there is one:



Q4. Find a hamiltonian cycle in the following graphs, if there is one:



Q5. Find a hamiltonian cycle in the following graph, if there is one:

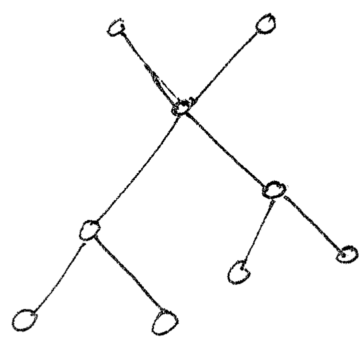


Notice that for graphs where each vertex has a high degree it was easy to find a hamiltonian cycle. It became harder and harder to find a hamiltonian cycle as the degrees of the vertices became less. In fact there is a result which says that if a connected graph has n vertices and the degree of each vertex more than n/2 then there is a Hamiltonian cycle.

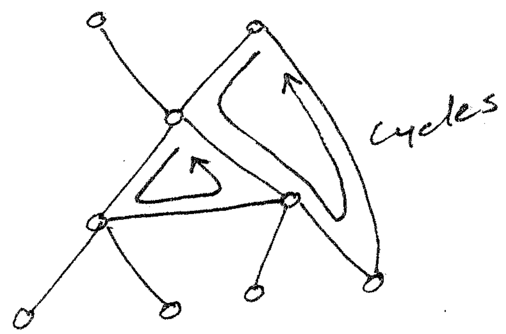
So, if each of your friends knew more than half of all the guest you invited to your party, then there is always a way to seat them in a circle so that each friend knows both his neighbors!

Section IV Trees and Minimal Spanning Trees

Definition: A *tree* is a connected graph with no cycles. Another way of saying this is that there is one and only one way to walk from any one vertex to any other. For example:

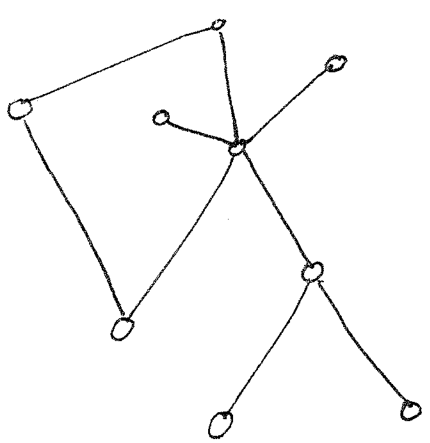


Is a Tree



Is NOT a tree

Q1. Is this graph a tree? Count number of vertices and edges.

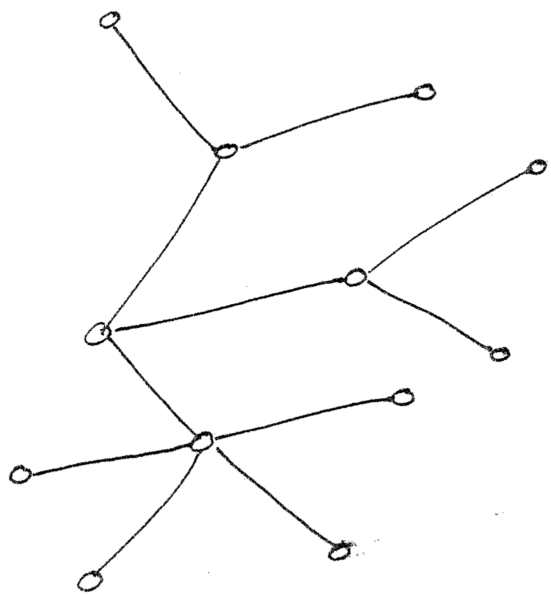


Not a tree

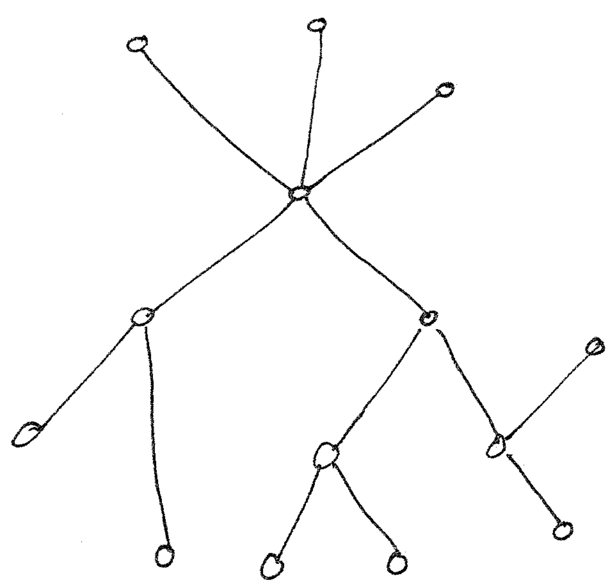
No. of Vertices : 9

No. of Edges : 9

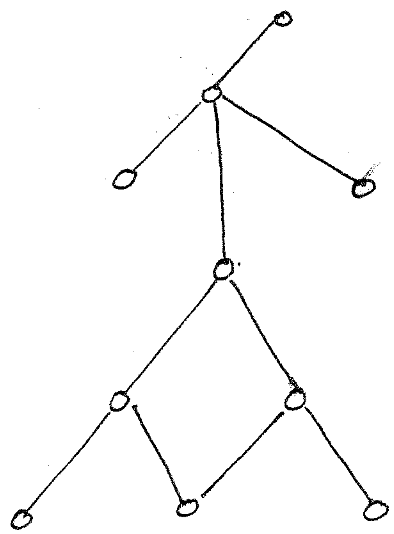
Q2. Is this graph a tree? Count number of vertices and edges.



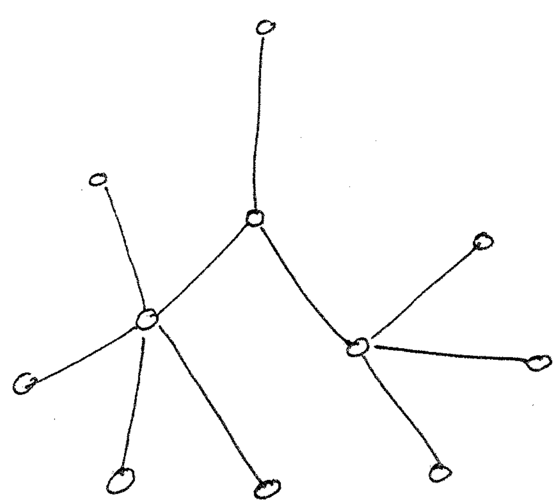
Q3. Is this graph a tree? Count number of vertices and edges.



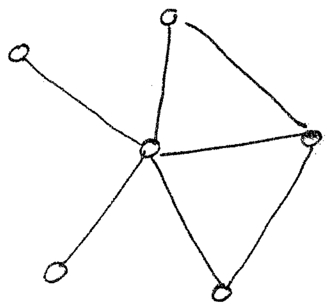
Q4. Is this graph a tree? Count number of vertices and edges.



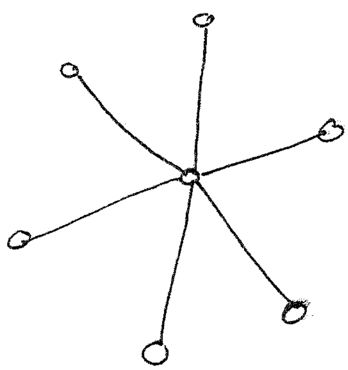
Q5. Is this graph a tree? Count number of vertices and edges.



Q6. Is this graph a tree? Count number of vertices and edges



Q7. Is this graph a tree? Count number of vertices and edges.



From the above 7 examples, do you see a pattern between the number of vertices and edges of the graph and when the graph is a tree? Discuss with your partner and write your answer below. Show it to a mentor for the next section.

Answer: A tree with n vertices has n-1 edges and any connected graph with n vertices and n-1 edges is in fact a tree.

How does that help us in a real-world situation. Try the following problems:

Q. Ten cities on a map are to be connected by building roads between the cities such that you can drive from any one city to the next. What is the minimum number of roads you would need to build?

Q. Thirteen computers are to be inter-connected such that you can send a message from any one of them any other. If there are eleven wires with you, can you connect the computers in such a way that you can send a message from any one of them to any other?

Q. An airline wants to build a network that connects a hundred cities in the US. To save costs, it wants to make sure that there is only one possible route (through various airports) to get from any one city to any other. What is the minimum number of flights that the airline needs to put in place?