The Fibonacci Sequence

Fibonacci (a.k.a. Leonardo of Pisa) asked the following problem in 1228. *A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced in a year, if the nature of these rabbits is such that every month each pair bears a new pair, a male and a female, which from their second month on, becomes productive?*

There are some implicit assumptions:

i) Rabbits never die or become infertile
ii) The first pair is newly born

Question 1. What is the answer to Fibonacci’s problem?
The Fibonacci sequence is defined this way.

\[
\begin{align*}
F_1 & = 1 \\
F_2 & = 1 \\
F_n & = F_{n-1} + F_{n-2}, \text{ for } n = 3, 4, 5, \ldots
\end{align*}
\]

Question 2. Calculate the first 12 Fibonacci numbers.

How many different ways can you climb a flight of 3 stairs, if you can go up either one stair or 2 stairs with each step?

Answer:

\[3 = 1 + 1 + 1 = 2 + 1 = 1 + 2,\]

so there are three ways.

Question 3: What if there were 5 stairs?

Question 4: What if there were 6 stairs?

Question 5: What if there were 11 stairs?

Question 6: What if there were \( n \) stairs? Why?
Choose any two positive whole numbers, $J_1$ and $J_2$ say (keep them small to make the arithmetic manageable).

Make your own Fibonacci sequence by inductively defining

$$J_n = J_{n-1} + J_{n-2}, \quad \text{for } n = 3, 4, 5, \ldots.$$ 

Question 7: What are the first 10 elements in your sequence?

Question 8. Add up the first 10 elements. Divide by 11. What do you notice?

Question 9: Do you think what you found depends on what two numbers you started with? Why?
Question 10. For $n = 2, 3, 4, \ldots$, calculate the ratio $\frac{F_n}{F_{n-1}}$. What do you notice?
Let us define a new sequence by the formula

\[ L_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}. \]

Question 11: What is \( L_1 \)?

Question 12: What is \( L_2 \)?

Question 13: What are \( L_3, L_4 \) and \( L_5 \)?

Question 14: What would you guess \( L_{12} \) is?
Question 15. It can be proved that \( L_n = F_n \) for every \( n \). What do you think the strategy would be to prove this?

\[
1 + \sqrt{5} = 3.236... \text{ and } 1 - \sqrt{5} = -1.236....
\]
So when \( n \) is large, \((1 + \sqrt{5})^n\) is much bigger than \(|(1 - \sqrt{5})^n|\). So a good approximation for \( L_n \) is

\[
A_n = \frac{(1 + \sqrt{5})^n}{2^n \sqrt{5}}.
\]

Question 16: What is \( \frac{A_n}{A_{n-1}} \)? How does this relate to Question 10?
Homework (voluntary)

Fibonacci numbers crop up frequently in nature. In particular, for many flowers the number of petals is a Fibonacci number. See e.g.
http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html#plants
for pictures of flowers and seed-heads, and a description of their relation
to Fibonacci numbers.

Exercise: Draw your own flower this way.
1. Put a dot in the center of a page. This is the center of the flower.
2. Choose some angle $x$.
3. Put a dot for your first petal a distance 1 away from the center.
4. Rotate by $x$, and put your second dot a distance $\sqrt{2}$ away.
5. Rotate by $x$, and put the third dot a distance $\sqrt{3}$ away.
6. Continue like this — the $n^{th}$ dot will be rotated $x$ from the previous
one, and be a distance $\sqrt{n}$ from the center.

If $x$ is 1/8 of a revolution — 45° — then the pattern is crowded at the
center, and spread out far away.

If $x$ is chosen as a Fibonacci ratio of revolutions — e.g. $\frac{21}{13}$ revolutions
which is 582° which is the same as 222° — the pattern looks much more like
a real daisy.