

Math Circle

03 March 2013

Even numbers are integers which can be written as one more than two times an integer, i.e. numbers of the form $2n + 1$ where n is an integer. For example, 1, 3, 5, 7, 9 are all odd numbers. Adding odd numbers together produces some surprising patterns.

Problem 1: A perfect square is an integer which can be written as another integer times itself, i.e. numbers of the form n^2 , where n is an integer. For example, 1, 4, 9, 16,... Can you figure out a way to write perfect squares as a sum of positive odd numbers? How do you know your pattern is correct for all perfect squares? Here are the first 21 odd numbers for reference:

1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41

For the last part, try thinking about the following picture:



Problem 2: Can you come up with a similar rule for fourth powers, i.e. $n^4 = (n^2)^2$? How about $n^6 = (n^3)^2$? How about any positive, even power, i.e. n^{2k} ? How do you know your pattern is correct for *all* even powers?

1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41

Problem 3: A perfect cube is an integer of the form n^3 for some integer. When n is a positive integer, can you come up with a pattern which describes n^3 as a sum of odd numbers? It might help to write out the first few positive perfect cubes. It's a little harder to see why your formula works, so we'll break that argument up into the next problems.

1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41

Problem 4: Can you come up with a formula to express $1 + 2 + 3 + \dots + n$? Try thinking about this array:

1	2	3	4	...	$n - 3$	$n - 2$	$n - 1$	n
n	$n - 1$	$n - 2$	$n - 3$...	4	3	2	1

Problem 5: If you want to compute n^3 , how many odd numbers do you have to skip? How many odd numbers do you have to add after that? Can you use #4 to prove that the pattern you derived in #3 is correct for all positive cubes?

Problem 6: Can you come up with a formula for $1^3 + 2^3 + 3^3 + \dots + n^3$? How many odd numbers are you adding together in this sum if you broke each cube into a sum of odds? How does this sum relate with the sum in #4?

Problem 7: Can you come up with a formula for $1^2 + 2^2 + 3^2 + \dots + n^2$? How does this sum relate to the following array? Can you change the order of the sums to compute the full sum?

1	+3	+5	+7	...	$+2n - 3$	$+2n - 1$
1	+3	+5	+7	...	$+2n - 3$	
...		
1	+3	+5	+7			
1	+3	+5				
1	+3					
1						

Problem 8: Can you come up with a formula for $1^4 + 2^4 + 3^4 + \dots + n^4$? Try making an array like in #7.

Problem 9: What should the coefficient of the highest degree term be in the formula for $1^p + \dots + n^p$? What does this tell you about the approximate value of $1^p + \dots + n^p$ when n is very large?

Problem 10: In problems #1-#5, we wrote perfect cubes and even powers as sums of *consecutive* odd numbers. What about higher odd powers of n , i.e. can we write n^5 , n^7 , and so on, in terms of consecutive odd numbers? We also wound up using all the odd numbers to express the cubes and even powers. Will we use all odd numbers in order to express them? Try finding the pattern for n^5 .