# Gaussian Numbers

**Integers**

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Addition of Gaussian integers

Examples:

\[(1 + i) + 1 = 2 + i\]
\[(1 + i) + i = 1 + 2i\]
\[(1 + i) + (1 + i) = 2 + 2i\]

1. Add \(1 + i\) and \(1 - i\).

2. Connect the points
(a) 0 and \(1 + i\);
(b) 0 and \(1 - i\);
(c) 0 and \((1 + i) + (1 - i)\);
3. Add $1 + 2i$ and 1.

4. Connect the points
   (a) 0 and $1 + 2i$;
   (b) 0 and 1;
   (c) 0 and $(1 + 2i) + 1$;
Multiplication of Gaussian Integers

Examples:

\[ 1 \times 1 = 1 \]
\[ 1 \times i = i \]

NOTE:
\[ i \times i = -1 \]

5. Multiply \( i \) and \( (1 + i) \)

6. Connect the points
   (a) 0 and \( 1 + i \);
   (b) 0 and \( i \times (1 + i) \);
7. Multiply $i$ and $(2 + i)$

8. Connect the points
   (a) 0 and $2 + i$;
   (b) 0 and $i \times (2 + i)$;
9. Multiply $1 + i$ and $(1 - i)$

10. Connect the points
(a) 0 and $1 + i$;
(b) 0 and $1 - i$;
(b) 0 and $(1 + i) \times (1 - i)$. 
11. Multiply $2 + i$ and $(2 - i)$

12. Connect the points
   (a) 0 and $2 + i$;
   (b) 0 and $2 - i$;
   (b) 0 and $(2 + i) \times (2 - i)$. 
13 Multiply $3 + 2i$ and $(3 - 2i)$

14. Connect the points
(a) 0 and $3 + 2i$;
(b) 0 and $3 - 2i$;
(b) 0 and $(3 + 2i) \times (3 - 2i)$. 
Prime Numbers

A positive integer $p$ is a prime number if it is divisible only by 1 and by itself.

15. List the prime numbers less than 30.

Square Numbers are

\begin{align*}
1 \times 1 &= 1 \\
2 \times 2 &= 4 \\
3 \times 3 &= 9 \\
&\vdots
\end{align*}

16. List the square numbers up to 200.
17. Which prime numbers can be written as a sum of two square numbers? (Hint: Try the prime numbers up to 30)
18. Compute the absolute value of $2 + i$ squared

$|2 + i|^2 = (2 + i)(2 - i)$.

How is this absolute value related to $2^2$ and $1^2$?

(b) Compute the absolute value of $3 + 2i$ squared

$|3 + 2i|^2 = (3 + 2i)(3 - 2i)$.

How is this absolute value related to $2^2$ and $3^2$?

(c) Compute the absolute value of $4 + i$ squared

$|4 + i|^2 = (4 + i)(4 - i)$.

How is this absolute value related to $4^2$ and $1^2$?)
19. Compute the products and their absolute values squared:

Multiply $1 + i$ and $2 + i$
Multiply $1 - i$ and $2 + i$
Multiply $2 + i$ and $2 + 3i$.
Multiply $2 - i$ and $2 + 3i$.
Multiply $2 + i$ and $4 + i$.
Multiply $2 - i$ and $4 + i$.
Multiply $3 + 2i$ and $4 + i$.
Multiply $3 - 2i$ and $4 + i$. 
20. For which two prime numbers can we express their product as a sum of two square numbers?

(Hint: Use questions 18 and 19. Try to express a product of two prime numbers less than 20 as a sum of two square numbers.)
21. In how many ways can we express the product of two prime numbers as a sum of two square numbers?
22. (Difficult problems) In how many ways can you express $5 \times 13 \times 17$ as a sum of two squares? How about $3 \times 5 \times 13$?