I. How would you find

\[ 1 + 2 + 3 + \cdots + 99 + 100? \]

II. How about

\[ 5 + 7 + 9 + \cdots + 83 + 85? \]

III. A general **Arithmetic Progression** with \( n \) terms is a sum of the form

\[ a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d). \]

The first term is \( a \), and \( d \) is called the common difference. Can you find a formula for (1)? Does it agree with what you got in I and II?
IV. What number corresponds to the binary number 11111111?

V. What number corresponds to the ternary (base 3) number 11111111?

VI. A general Geometric Progression with $n$ terms is a sum of the form
\[ a + ar + ar^2 + \cdots + ar^{n-1}. \] (2)

Here, $a$ is the first term, and $r$ is called the common ratio. Can you find a formula for (2)? (Hint: what happens if you multiply (2) by $r$?)

Does the formula agree with what you got in IV and V?
Suppose we want to find
\[1^2 + 2^2 + \cdots + 100^2.\]
This is a bit trickier. We would like a formula for
\[1^2 + 2^2 + \cdots + n^2. \tag{3}\]
How do we find it? We are going to start by revisiting III to come up with an algebraic way of finding
\[1 + 2 + \cdots + n, \tag{4}\]
and then see if we can generalize this to sums of squares. Draw a picture of (4):

We may guess that the order of magnitude is about \(n^2\) (it is close to a triangle with base and height \(n\), but the diagonal is jagged). Let us guess that
\[1 + 2 + \cdots + n = An^2 + Bn + C, \tag{5}\]
where \(A\) is about \(\frac{1}{2}\), and \(Bn + C\) is a correction term to make up for the jagged edge. We have to find what \(A\), \(B\) and \(C\) are. First, we rewrite (5) as
\[0 + 1 + 2 + \cdots + n = An^2 + Bn + C; \tag{6}\]
and let \(n = 0\); this tells us \(C = 0\). Next, we add \(n + 1\) to both sides of (5) or (6):
\[1 + 2 + \cdots + n + (n + 1) = An^2 + Bn + (n + 1) = A(n + 1)^2 + B(n + 1).\]
Expanding $(n + 1)^2 = n^2 + 2n + 1$, we get
\[ An^2 + (B + 1)n + 1 = An^2 + (2A + B)n + (A + B). \] (7)

Comparing coefficients on both sides of (7), we get
\[ B + 1 = 2A + B \]
\[ 1 = A + B \]

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{2}$, and hence
\[ 1 + 2 + \cdots + n = \frac{1}{2}n^2 + \frac{1}{2}n. \] (8)

This is a more cumbersome way to find (8) than what we did in III, but the method generalizes to give a formula for (3). We start by assuming
\[ 1^2 + 2^2 + \cdots + n^2 = An^3 + Bn^2 + Cn. \] (9)

Now add $(n + 1)^2$ to both sides of (9). Combine coefficients of powers of $n$, and you get 3 equations in $A$, $B$ and $C$. The nice thing is that these equations are triangular, so easy to solve. What equations do you get? What is their solution?

VII. What is the formula for
\[ 1^2 + 2^2 + \cdots + n^2? \]
VIII. Can you find a formula for

\[ 1^3 + 2^3 + \cdots + n^3? \]
IX. How do we know our formulas from VII and VIII are correct?