Summing Series

John E. McCarthy

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I. How would you find

$$1 + 2 + 3 + \cdots + 99 + 100?$$

II. How about

$$5 + 7 + 9 + \cdots + 83 + 85?$$

III. A general *Arithmetic Progression* with n terms is a sum of the form

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d). \tag{1}$$

The first term is a, and d is called the common difference. Can you find a formula for (1)? Does it agree with what you got in I and II?

IV. What number corresponds to the binary number 11111111?

V. What number corresponds to the ternary (base 3) number 11111111?

VI. A general *Geometric Progression* with n terms is a sum of the form

$$a + ar + ar^2 + \cdots + ar^{n-1}. \tag{2}$$

Here, a is the first term, and r is called the common ratio. Can you find a formula for (2)? (Hint: what happens if you multiply (2) by r?)

Does the formula agree with what you got in IV and V?

Suppose we want to find

$$1^2 + 2^2 + \cdots + 100^2.$$

This is a bit trickier. We would like a formula for

$$1^2 + 2^2 + \cdots + n^2. \tag{3}$$

How do we find it? We are going to start by revisiting III to come up with an algebraic way of finding

$$1 + 2 + \cdots + n, \tag{4}$$

and then see if we can generalize this to sums of squares. Draw a picture of (4):

We may guess that the order of magnitude is about n^2 (it is close to a triangle with base and height n, but the diagonal is jagged). Let us guess that

$$1 + 2 + \cdots + n \;=\; An^2 + Bn + C, \tag{5}$$

where A is about $\frac{1}{2}$, and $Bn + C$ is a correction term to make up for the jagged edge. We have to find what A, B and C are. First, we rewrite (5) as

$$0 + 1 + 2 + \cdots + n \;=\; An^2 + Bn + C, \tag{6}$$

and let $n = 0$; this tells us $C = 0$. Next, we add $n + 1$ to both sides of (5) or (6):

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= An^2 + Bn + (n+1) \\ &= A(n+1)^2 + B(n+1). \end{aligned}$$

Expanding $(n+1)^2 = n^2 + 2n + 1$, we get

$$An^2 + (B+1)n + 1 \;=\; An^2 + (2A+B)n + (A+B). \tag{7}$$

Comparing coefficients on both sides of (7), we get

$$\begin{aligned} B+1 &= 2A+B \\ 1 &= A+B \end{aligned}$$

Solving, we get $A = \frac{1}{2},\ B = \frac{1}{2}$, and hence

$$1 + 2 + \cdots + n \;=\; \frac{1}{2}n^2 + \frac{1}{2}n. \tag{8}$$

This is a more cumbersome way to find (8) than what we did in III, but the method generalizes to give a formula for (3). We start by assuming

$$1^2 + 2^2 + \cdots + n^2 \;=\; An^3 + Bn^2 + Cn. \tag{9}$$

Now add $(n+1)^2$ to both sides of (9). Combine coefficients of powers of n, and you get 3 equations in A, B and C. The nice thing is that these equations are triangular, so easy to solve. What equations do you get? What is their solution?

VII. What is the formula for

$$1^2 + 2^2 + \cdots + n^2?$$

VIII. Can you find a formula for

$$1^3 + 2^3 + \cdots + n^3?$$

IX. How do we know our formulas from VII and VIII are correct?