Mathematical Black Holes

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1 The Collatz Conjecture

King Minos has just imprisoned Daedalus and his son Icarus in a high tower. Daedalus was implicated in the murder of the Minotaur, who happened to be King Mino's son.

Daedalus invents a way to escape. He and Icarus gather bee's wax and birds feathers to make wings to fly off the tower. The night before the flight, Icarus has a dream:

> He write a number on a rock, and he throws the rock off the tower. If the number on the rock is even, then it is halved. If the number on the rock is odd, then its tripled and one is added. Icarus continues this and if he ends up with 1 then he crashes into the sea.

Icarus knows that if he can do this and end up with something other than 1, then he will survive.

Daedalus has a similar dream, with one difference:

> He write a number on a rock, and he throws the rock off the tower. If the number on the rock is even, then it is halved. If the number on the rock is odd, then its tripled and one is subtracted. Icarus continues this and if he ends up with 1 then he crashes into the sea.

The task is to save Icarus and Daedalus.

Notes: Run it with starting number of 15

1. Try to save Icarus. Pick a starting number and run Icarus' dream over and over. If you eventually get to 1, Icarus dies. If you never get to 1, Icarus lives.
2. Try to save Daedalus. Pick a starting number and run Icarus' dream over and over. If you eventually get to 1, Icarus dies. If you never get to 1, Icarus lives.

Don't be discouraged if you were unable to save Icarus. The Collatz conjecture was first proposed by Lothar Collatz in 1937. Collatz conjectured that the Icarus sequence must always end in a 1. The conjecture has been tested by a computer, for all starting values up to $5 \times 2^{60} \approx 5.764 \times 10^{18}$.

Every time you tried to save Icarus, you probably got to the loop $4 \to 2 \to 1 \to 4 \to 2 \to 1 \to 4 \to \cdots$.

3. Is $4 \to 2 \to 1 \to 4 \to \cdots$ the only Icarus loop possible?

   Solution: This is the Collatz Conjecture and the answer is unknown.

4. Find as many different cycles that save Daedalus. (There are several.)

   Solution: Here are all the possible cycles I found with starting numbers less than 1,000,000.

Cycle Length	**Starting Numbers**
2	1, 2
5	5, 7, 10, 14, 20
18	17, 25, 34, 37, 41, 50, 55, 61, 68, 74, 82, 91, 110, 122, 136, 164, 182, 272

Lets investigate the length of Icarus descents–how long does it take from your starting number to get to the number 1. For example, if we start with 5:

$$5 \to 16 \to 8 \to 4 \to 2 \to 1$$

and this descent has length 6.

5. Find the lengths of descents of all the numbers:

Number	**Length of Descent**
1	1
2	2
3	8
4	3
5	6
6	9
7	17
8	4
9	20
10	7
11	15
12	10
13	10
14	18
15	18
16	5
17	13
18	21
19	21
20	8

6. Is every descent length possible?

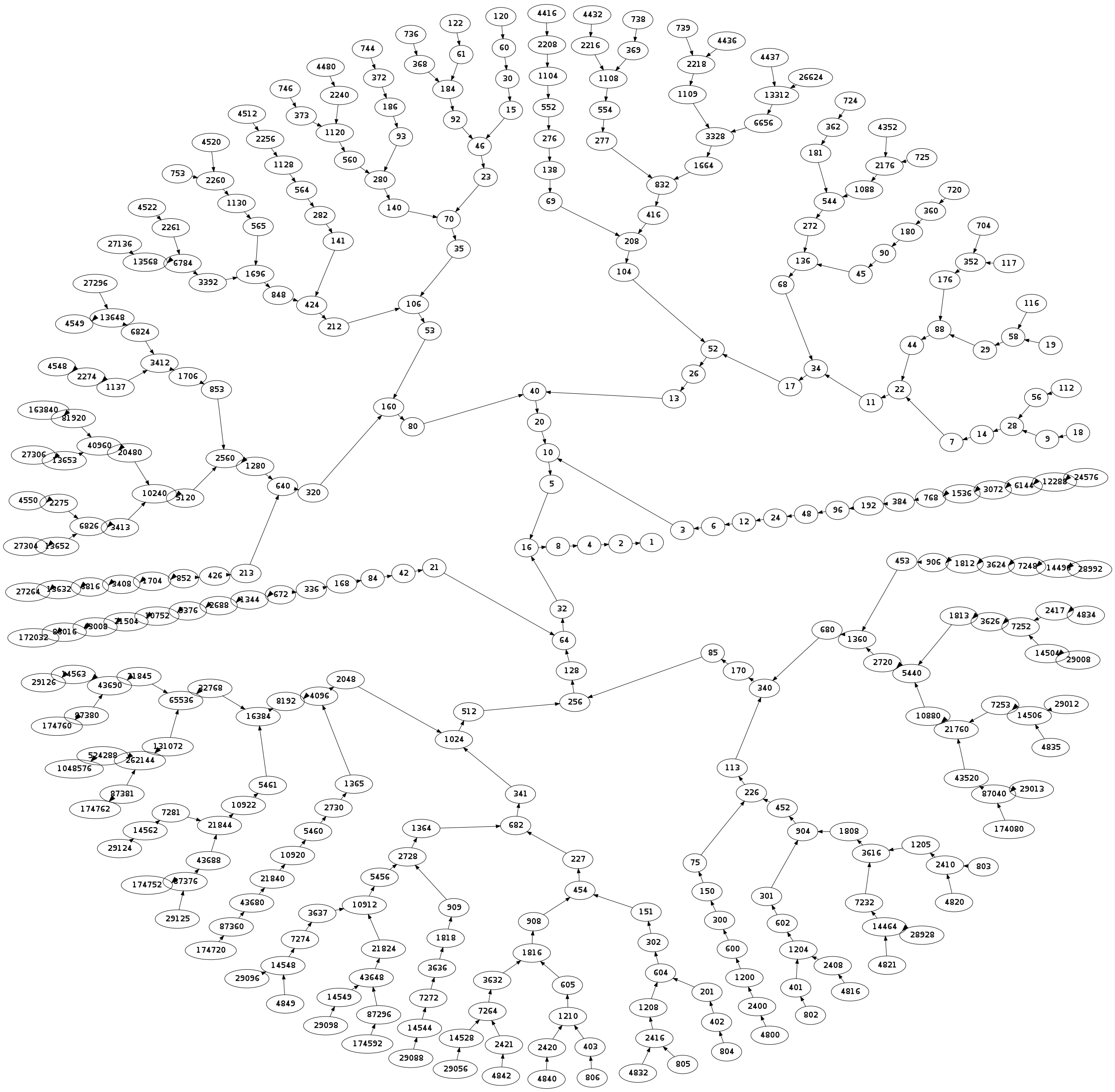
Here is some data for data for starting numbers less than 100. The longest length for a starting number less than 1,000,000 I found was a length of 525 for number 837,779.

Length of Descent	Starting Numbers
1	1
2	2
3	4
4	8
5	16
6	5, 32
7	10, 64
8	3, 20, 21
9	6, 40, 42
10	12, 13, 80, 84, 85
11	24, 26
12	48, 52, 53
13	17, 96
14	34, 35
15	11, 68, 69, 70, 75
16	22, 23
17	7, 44, 45, 46
18	14, 15, 88, 90, 92, 93
19	28, 29, 30
20	9, 56, 58, 60, 61
21	18, 19
22	36, 37, 38
23	72, 74, 76, 77, 81

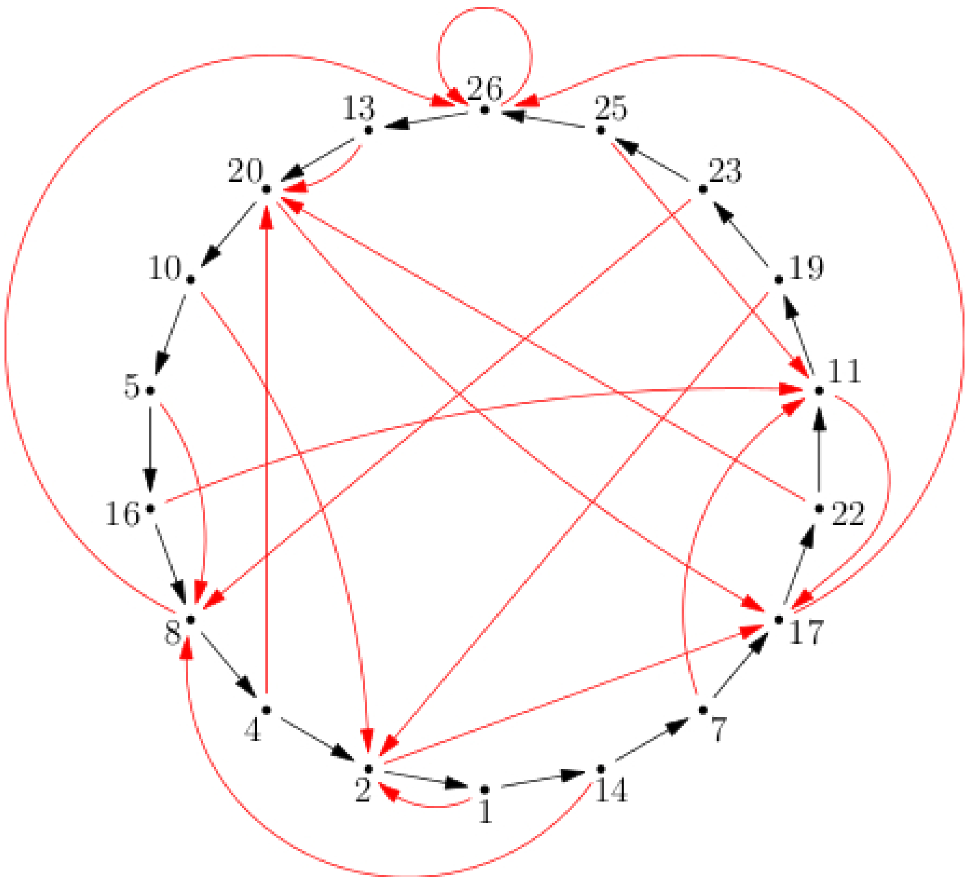
Length of Descent	Starting Numbers
24	25
25	49, 50, 51
26	98, 99, 100
27	33
28	65, 66, 67
30	43
31	86, 87, 89
33	57, 59
35	39
36	78, 79
93	91
103	71
105	47
106	94, 95
107	31
108	62, 63
110	41
111	82, 83
112	27
113	54, 55
116	73
119	97

7. Is every descent length possible?

You could also draw the Collatz graph:



Or the Collatz graph, mod 27:



8. Experiment with other Collatz-like conjectures. Use $5x + 1$ instead of $3x + 1$ and see what happens. Make up other generalizations.

   Solution: For the $5x + 1$, I found cycles for starting numbers $1, 2, 3, 4, 5, 6$ but for $n = 7$, even with $100,000$ iterations, it did not appear to repeat.

2 Other Mathematical Black Holes

A mathematical black hole is something like the Collatz Conjecture above. Something that seems to suck all the numbers into one number (so, everything was sucked into 1 for the Collatz Conjecture).

See if you can show that the following are black holes. Prove that they are black holes–that they always end at the same place.

9. Start with any natural number, such as 9288759. Count the number of even digits, the number of odd digits and the total number of digits. Form a new number from this three numbers and repeat. Keep repeating:

   $$9288759 \rightarrow 347 \rightarrow \cdots$$

   What happens? Be sure to try it with some really big numbers.

Solution: You always end up with 123.

Here are the keys to proving this is a black hole: Let $f(n)$ be the function here.

- $f(123) = 123$.
- For $n > 999$, $f(n) < n$. Can prove directly.
- For $n < 1000$, $f(n)$ can only be one of the following: 033, 123, 213, 303.

10. Start with any whole number, write out its number in English, count the number of characters in the spelling to get a new number and repeat. What happens?

    For example:

    $$163 \longrightarrow \text{one hundred sixty three} \longrightarrow 23 \longrightarrow \text{etc.}$$

    Try it with another language if you know another language.

    Solution: You always end up with 4.

11. Start with positive multiple of 3. Write down your multiple of 3 and, one at a time, take the cube of each digit. Add up the cubes to form a new number and repeat.

    For example:

    $$24 \to 2^3 + 4^3 = 72 \to \cdots$$

    What if you don't have a multiple of 3?
    What about for 4-th or 5-th powers?

    Solution: You always end up with 153.

    If you don't start with a multiple of 3, you will always get a cycle. Similarly with other powers.

    Why do powers of 3 always end with 153?

    It is known that the only numbers equal to the sum of the cubes of their digits are 153, 370, 371 and 407.

    In this case for large n, $f(n) < n$.

12. Take any 4-digit number that does not have all 4 digits identical. Rearrange the digits to form the largest and the smallest numbers possible and subtract the two numbers. Repeat.

    What happens if you start with a different number of digits (3, 5, 6, etc.)?

    For example:

    $$1423 \to 4321 - 1234 = 3087 \to \cdots$$

    Solution: If you start with a 4 digit number, you always end up with 6174, If you start with a 3 digit number, you always end up with 495, which is known as Kaprekar's constant.

    Note that for 4 digit numbers, it always takes at most 7 steps.

13. Take any natural number larger than 1 and write down all its divisors including 1 and the number itself. Take the sum of these digits and repeat.

    Solution: End up with 15

14. Take an integer and find its largest odd factor. Triple it and add one. Repeat.

    So, for 84 we would get $84 \to 21 \to 64 \to \cdots$.

    Solution: You should always get to 4 and it is equivalent to the Collatz conjecture.

3 Black Holes and Real Numbers

All of the previous problems are concerned with functions on natural numbers. Here we are going to look at fixed points for functions or sequences of real numbers. There are many important theorems about fixed points and iteration. This same idea is key in finding roots of equations using Newton's Method from calculus.

15. Take your scientific calculator (make sure your calculator is in radian mode). Start with $x = 1$ and iterate the cosine function (keep pushing the cosine button). What happens and why?

    Solution: The solution to $\cos x = x$ is approximately 0.739085 radians.

16. Let $f(x) = (x + 3/x)/2$. Start with $x = 1$ and iterate. What happens and why?

    What happens if you start with $x = 2$? Wha happanes if you start with $x = -2$?

    Solution: $f^n(1) \longrightarrow \sqrt{3}$

17. The Fibonocci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$. Let F_n be the n-th Fibonocci number. So, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, etc. Note that $F_n = F_{n-1} + F_{n-2}$.

    For each value of n, compute f_{n+1}/f_n. I did the first several for you:

n	F_{n+1}/F_n
1	1
2	3/2
3	
4	
5	
6	
7	
8	
9	
10	

    What happens as you iterate more and more? Can you prove your result?

    Solution: One way to convince yourself the limit is $\phi = (1 + \sqrt{5})/2$ is to assume the that the limit exists, L. And take $F_n = F_{n-1} + F_{n-2}$ and divide by F_{n-1}. For large n, this is the same as $L = 1 + 1/L$, or $L^2 - L - 1 = 0$. Solving gives ϕ.

18. Starting with $x = 0$, iterate $f(x) = e^{-x}$ (use a calculator).
    What happens for different starting numbers?

    Solution: Solves $x = e^{-x}$ which is approximately 0.567143290409

19. Take a 4 function calculator (or, pretend your calculator only has 4 functions: $+$, $-$, $\times$, $\div$). Come up with an iterative functions that will give you $\sqrt{2}$.

    Solution: You can use $f(x) = \frac{1}{2}(x + 2/x)$

20. Can you find a different iterative function that will also give you $\sqrt{2}$?

    Solution: What I did was, for any k:

$$\begin{aligned} x^2 &= 2 \\ x &= \frac{2}{x} \\ x + kx &= kx + \frac{2}{x} \\ x &= \frac{1}{k+1}\left(kx + \frac{2}{x}\right) \end{aligned}$$

21. Can you find an iterative function that will give you $\sqrt{n}$ for any n?

    Solution: You can use $f(x) = \frac{1}{2}(x + n/x)$

References

[1] Ecker, Michael. "Number Play, Calculators and Card Tricks: Mathemagical Black Holes." From "A Tribute to Martin Gardner."

[2] Hamilton, Gordon. *The MathPickle*, http://www.mathpickle.com.