

Pascal's Triangle and Fractals!

Washington University Math Circle

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Pascal's triangle is a famous mathematical structure because of its beauty and usefulness. Pascal's triangle is named after the French mathematical prodigy Blaise Pascal (1623-1662). In addition to work in theoretical mathematics, Pascal worked in physics and philosophy, was a writer, and was one of the first inventors of the mechanical calculator. Although Pascal's triangle is not difficult to construct, it has many complex patterns hidden inside of it.

Rules of Pascal's Triangle

1. Put a 1 at the top of your page
2. Start and end each new row with a 1
3. Continue placing numbers below in a triangular pattern so that each new number is the sum of the two above it.

Activity: Create Pascal's triangle on the next page.

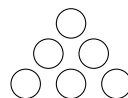
Pascal's Triangle

The Natural Numbers: The natural numbers (sometimes written \mathbb{N}) are the numbers

$$1, 2, 3, 4, \dots$$

Activity: Find the natural numbers in Pascal's triangle. Can you use the rules for creating Pascal's triangle to explain why this pattern occurs?

The Triangular Numbers: The triangular numbers come from counting the number of dots in equilateral triangles that look like:



Activity: Find the first 10 triangular numbers.

Challenge Problem: Let T_n be the n^{th} triangular number. Show that

$$T_n = 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

(Hint: Convince yourself that $T_n = 1 + 2 + \dots + n$ using the pictures of the triangular numbers. Look up “mathematical induction” to show $1 + 2 + \dots + n = \frac{n(n+1)}{2}$.)

Activity: Find the triangular numbers in Pascal's triangle. Can you use the rules for creating Pascal's triangle to explain why this pattern occurs?

The powers of 2: The powers of 2 are the numbers of the form

$$2^0, 2^1, 2^2, 2^3, \dots$$

(Remember $2^n = 2 \times 2 \times \dots \times 2$ n times)

Activity: Find the first 10 powers of 2.

Activity: Find the powers of 2 in Pascal's triangle. (Hint: You will have to combine numbers in Pascal's triangle to find the pattern.)

The Fibonacci Numbers: To get the Fibonacci numbers, start with the numbers 0 and 1. Get the next number by adding the previous two numbers. This can be written

$$F_n = F_{n-1} + F_{n-2} \quad F_0 = 0, F_1 = 1$$

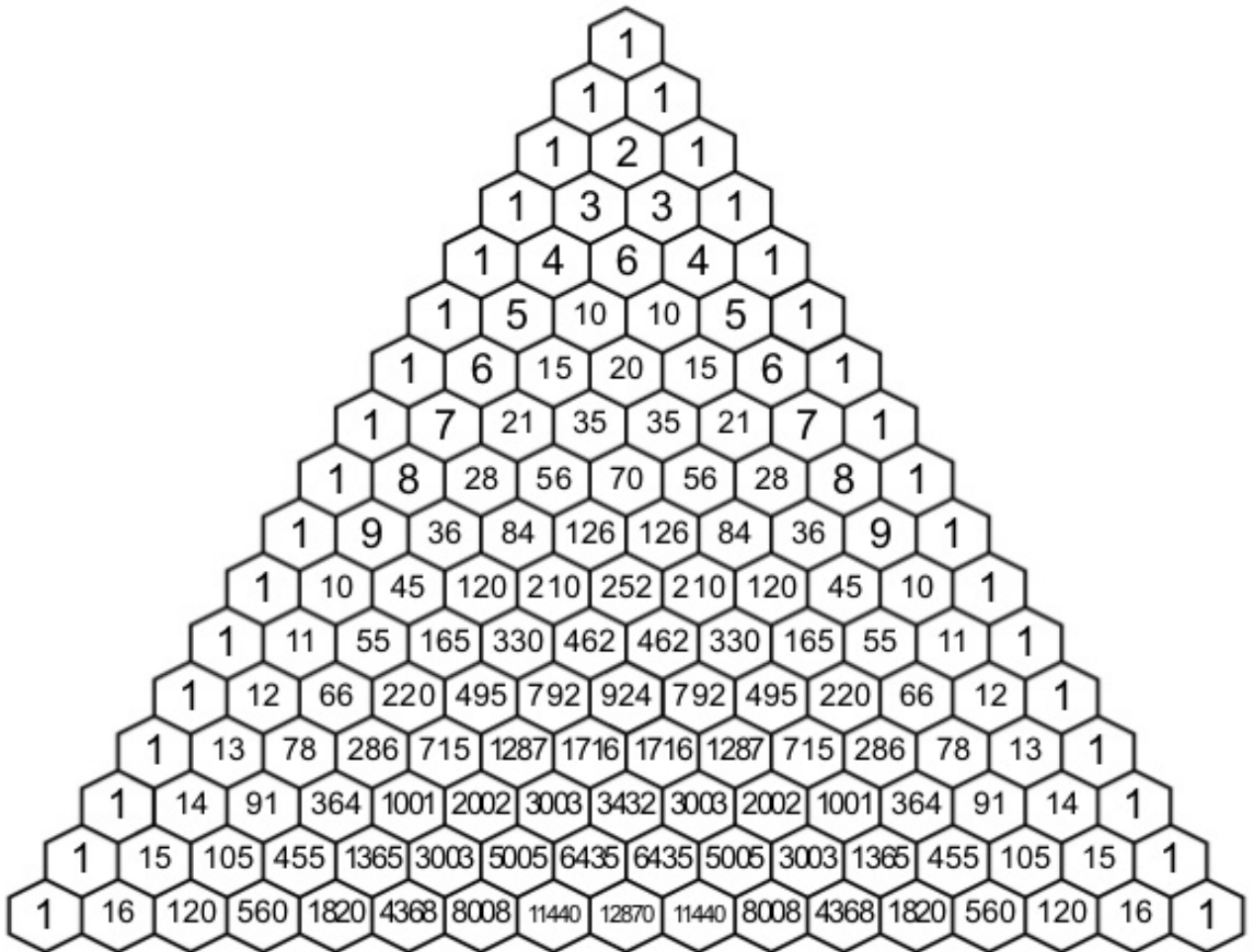
where F_n is the n^{th} Fibonacci number.

Activity: Find the first 10 Fibonacci numbers.

Activity: Find the Fibonacci numbers in Pascal's triangle.

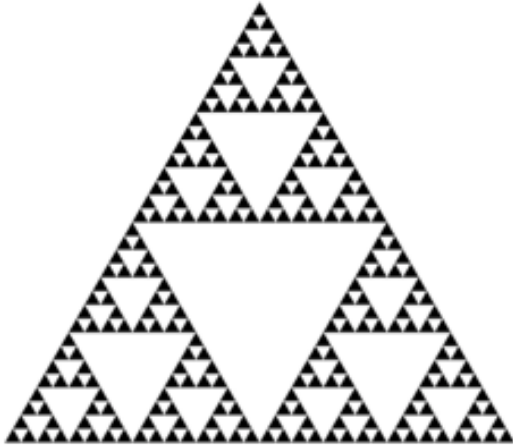
The last four examples illustrate that some interesting sequences of numbers are encoded by Pascal's triangle. The next activity shows that interesting geometric patterns are encoded by Pascal's triangle as well.

Activity: Shade in all of the odd numbers in Pascal's triangle.

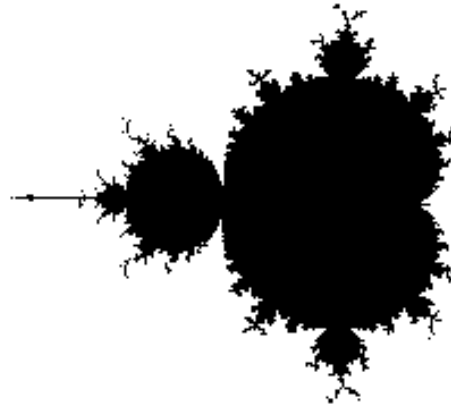


This shape is an example of a geometric object called a fractal. Fractals exhibit a property called self-similarity. This means that as you zoom in on the fractal, you see the same shapes and patterns. The term fractal was introduced in 1975 by Benoit Mandelbrot– they were studied very recently. Check out these cool fractals!

Sierpinski's Triangle



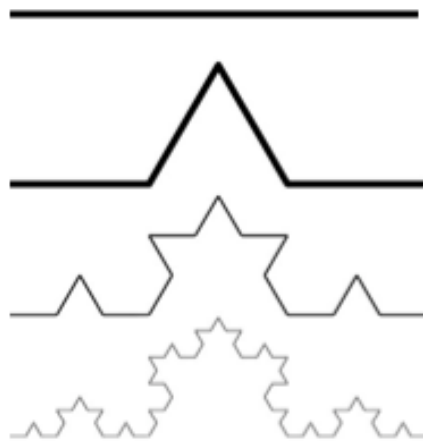
Mandelbrot Set



Cantor Set



Koch Curve



Activity: Calculate the lengths of the first 3 iterations of the Koch curve (assume the length of the first iteration is 1).

Question: What is the length of the n^{th} iteration of the Koch curve?

Question: What happens to these lengths as the number of iterations increases?

This shows that there is a curve inside a square with sides of length 1 that has infinite length!

More things to think about!

Question: Can you find a pattern for the even numbers in the Fibonacci sequence? Explain why this pattern occurs.

Question: You go trick-or-treating at a house where you are offered 2 different types of candy. You may choose to take 0 or 1 pieces of each type of candy. How many different arrangements of candy could you have after leaving this house? What if they offered 3 different candies? What if they offered 4 different candies?

Question: The triangular numbers count the dots in 2-dimensional equilateral triangles. Can you define a sequence of numbers that counts the dots in 3-dimensional regular pyramids (with triangular bases)? Write out the first 8 of these numbers. These are called the tetrahedral numbers. Here's a picture to help.

