Dots, Lines, and Graphs

Washington University Math Circle

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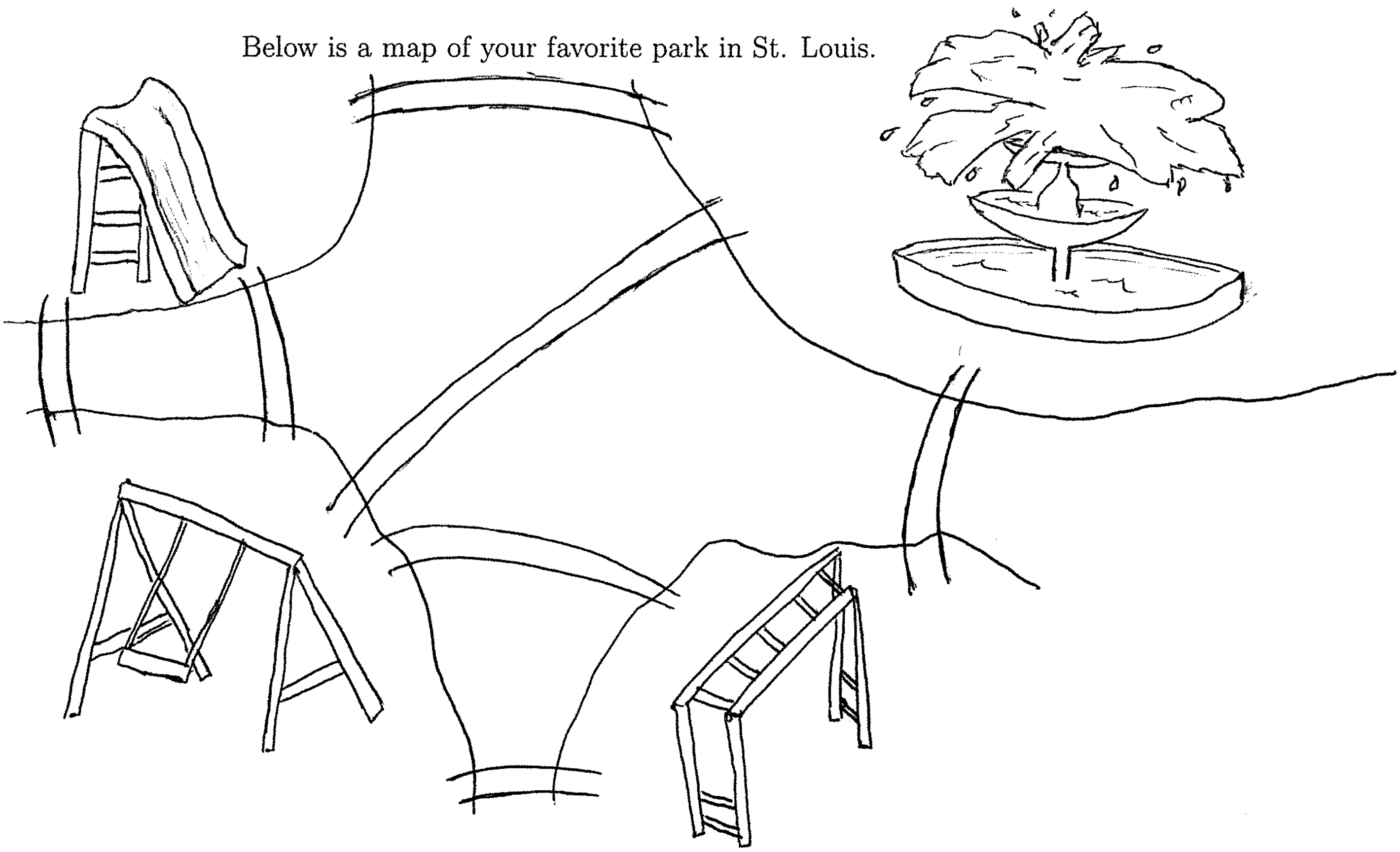
Below is a map of your favorite park in St. Louis.



Like the curious mathematician you are, you wonder if you could walk in a path that starts at the slide, visits each attraction, and crosses each bridge exactly one time.

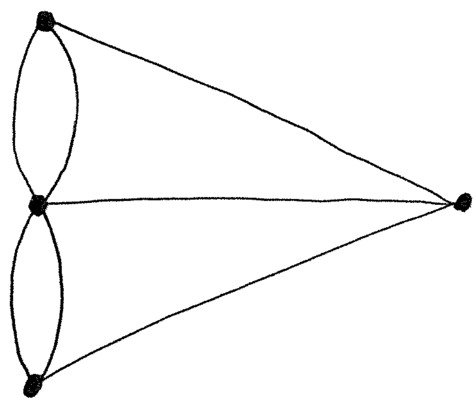
Activity: Find a path through the park that starts at the slide, ends at the slide, and crosses each bridge exactly one time.

If you're having trouble, try the same thing except starting and ending at the swings, the monkey bars, or the fountain.

Still having trouble? Maybe it is not possible to walk such a path.

Activity: Explain why it would not be possible to walk a path as described in the first activity.

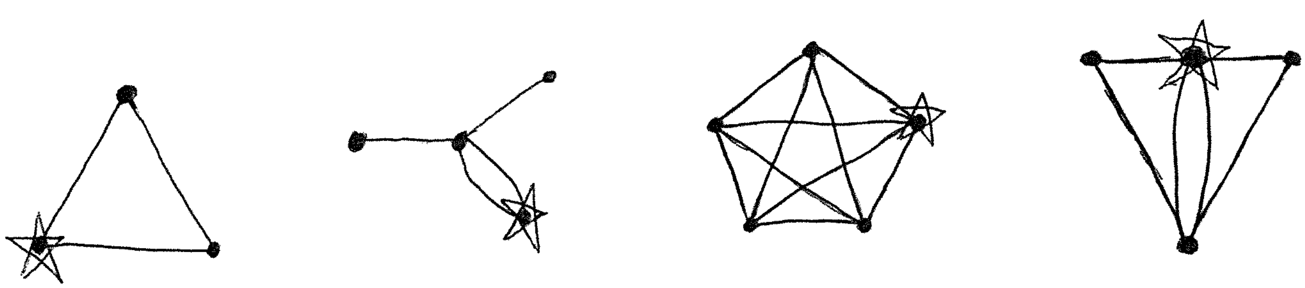
Let's make this more mathematical. Here is a simpler representation of the map.



The dots (vertices) represent the attractions, and the lines (edges) represent the bridges. In this context, we want to start at a vertex, follow edges in a path (using each edge only once), and end at the same vertex. If you have not completed either of the previous activities, ask a mentor for help.

This is a famous problem known as the "Seven Bridges of Königsberg." The city of Königsberg (located in present day Russia) also has 4 land masses separated by the Pregel River and 7 bridges arranged in the same way as the park. The famous mathematician Leonhard Euler tried to solve this puzzle, and with his solution he developed the field of math known as Graph Theory!

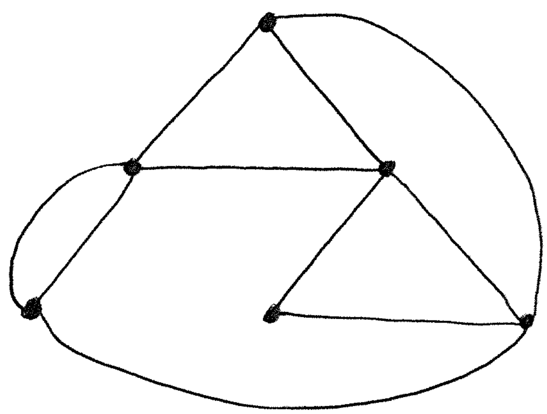
A graph is a collection of vertices and edges connecting these vertices. Here are some graphs.



Activity: For each of graphs at the bottom of the previous page, try to find a path that starts and ends at the starred vertex and crosses each edge exactly once.

In his solution to the Seven Bridges of Königsberg problem, Euler explained that it is possible to walk one of our desired paths exactly when each vertex in the graph representation has an even number of edges coming from it. Can you see why this is true?

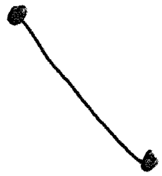
Let's look at other properties of graphs. Here's a graph.

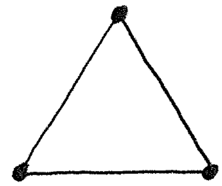


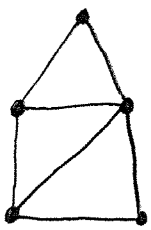
Activity: Count the number of vertices, edges, and regions this graph splits your paper into.

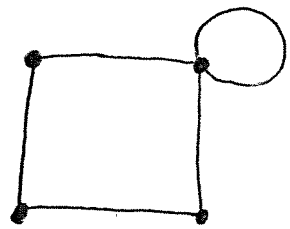
Euler also counted these numbers for many different graphs and he found an interesting pattern.

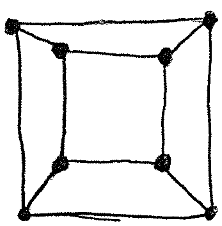
Activity: Count the number of vertices, edges, and regions the graph splits your paper into for each of the graphs on the next page.

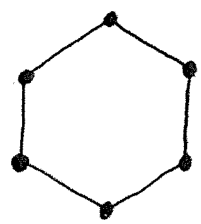


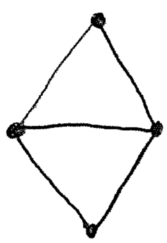


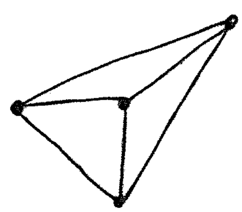










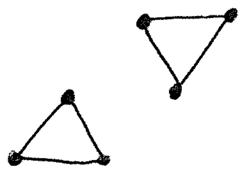


Can you notice a pattern with these numbers among the graphs?

Hint: Calculate

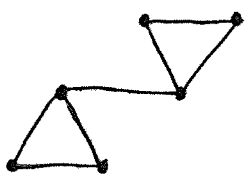
$$(\# \text{ Vertices}) - (\# \text{ Edges}) + (\# \text{ Regions})\,.$$

Did you find the pattern? What about this graph?



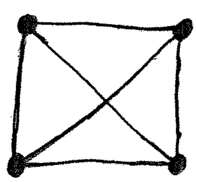
$(\# \text{ Vertices}) - (\# \text{ Edges}) + (\# \text{ Regions}) =$

But this graph is not connected. Add an edge to make it connected and try again.



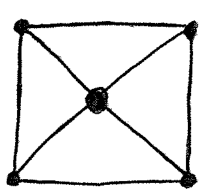
$(\# \text{ Vertices}) - (\# \text{ Edges}) + (\# \text{ Regions}) =$

Or what about this graph?



$(\# \text{ Vertices}) - (\# \text{ Edges}) + (\# \text{ Regions}) =$

But this graph has two edges that cross each other. Add a vertex to fix this and try again.



$(\# \text{ Vertices}) - (\# \text{ Edges}) + (\# \text{ Regions}) =$

Still don't believe the pattern?

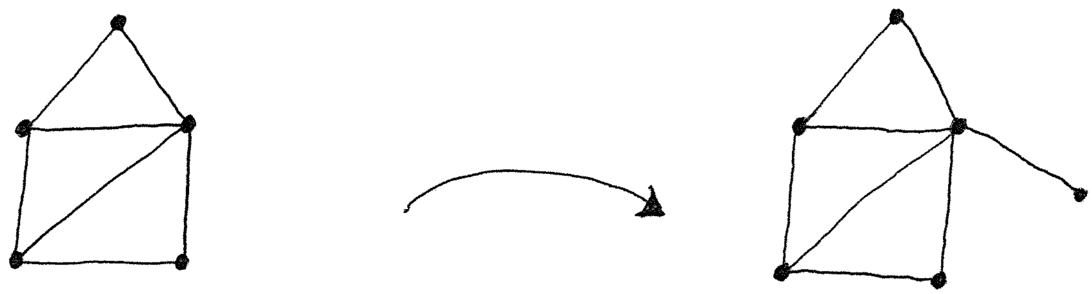
Activity: Make your own graphs and test it out! Just make sure your graphs are connected and that no two edges cross each other.

Why is (# Vertices) − (# Edges) + (# Regions) always equal to 2 for connected graphs with non-intersecting edges?

It's true for the simplest graph – a single vertex.

(# Vertices) − (# Edges) + (# Regions) =

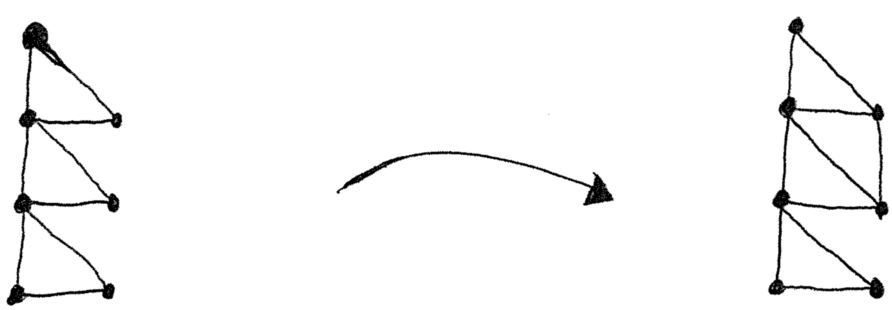
Adding a vertex connected to a graph with an edge does not change (# Vertices) − (# Edges) + (# Regions).



(# Vertices)−(# Edges)+(# Regions) (# Vertices)−(# Edges)+(# Regions)

= =

Adding an edge connecting two already existing vertices also does not change (# Vertices)− (# Edges) + (# Regions).



(# Vertices)−(# Edges)+(# Regions) (# Vertices)−(# Edges)+(# Regions)

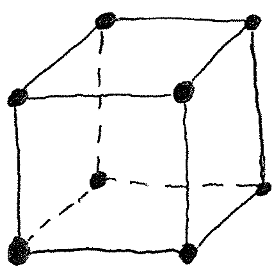
= =

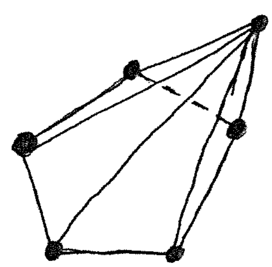
Since any connected graph with non-intersecting edges can be drawn by first drawing a single vertex and then adding edges and a new vertex or adding edges between existing vertices, (# Vertices) − (# Edges) + (# Regions) is the same for all of these graphs!

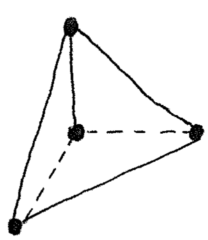
The number (# Vertices) − (# Edges) + (# Regions) is called the "Euler characteristic" of a graph. We just argued that the Euler characteristic is equal to 2 for every connected planar graph. This argument contains the ideas behind a proof by "mathematical induction." Mathematical induction is an extremely useful tool in mathematics. It basically says that to argue a statement is true, you can show that the statement is true in its most simple case and then show that it remains true as you make the statement one step more complicated.

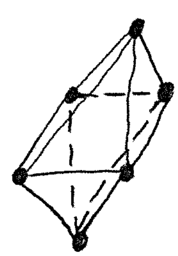
A similar pattern can be noticed in a higher dimensional scenerio.

Activity: Calculate (# Vertices) − (# Edges) + (# Faces) for these 3D solids.









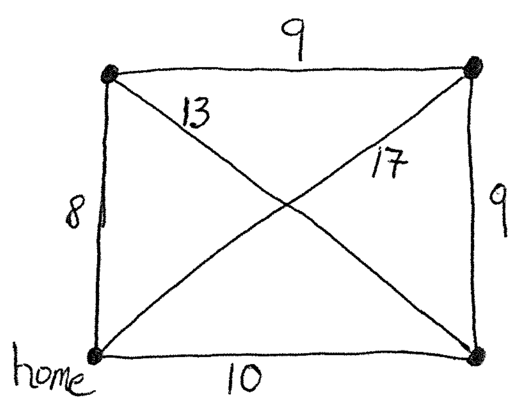
The Traveling Salesman Problem

Graph theory is still being investigated by mathematicians today. The traveling salesman problem is one of the most famous unsolved problems related to graph theory. If the traveling salesman problem problem would be solved, not only would this be a huge discovery in mathematics and computer science, but it would lead to many life changing applications in the real world.

Problem: Suppose that you are a door-to-door salesperson who needs to make 50 stops in your day of work. Since you want to be home from work in time for dinner, you would like to find the shortest path that starts at your house, visits each location, and ends back at your house.

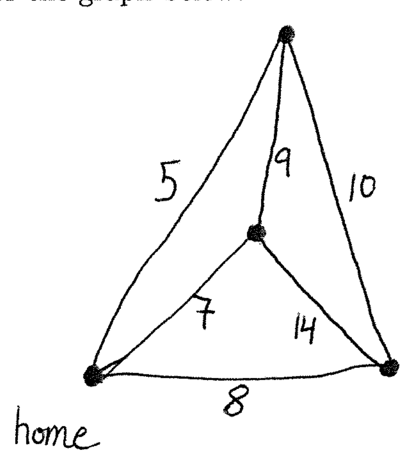
Represent the locations with vertices of a graph and label the edges by the distances between the locations to think of this as a graph theory problem.

Activity: Find the shortest path through each vertex that starts and ends at home for the graph below.



One idea is to choose the path that follows the edge of smallest distance at each step. This "solution" is called a greedy algorithm.

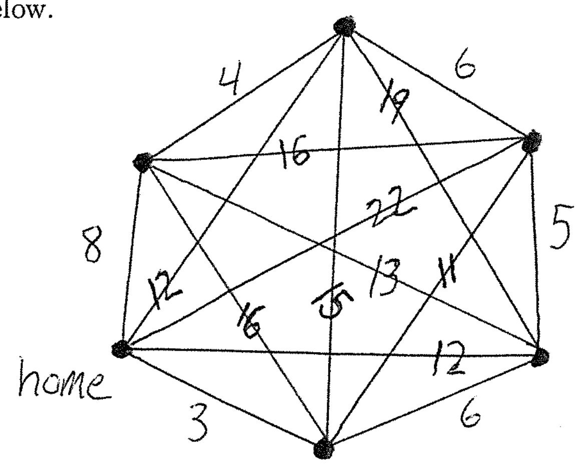
Activity: Use a greedy algorithm to find a path through each vertex that starts and ends at home for the graph below.



This is an easy way to find a path that visits all destinations, but it does not give the shortest path for this graph. Can you find a shorter path that does not use a greedy algorithm?

Another idea is to calculate the distance of every possible path you could take. This is called a brute force algorithm.

Activity: Use a brute force algorithm to find the shortest path that starts and ends at home for the graph below.



Don't work too hard on this activity... you have 5 choices when you leave home, 4 choices after your first destination, 3 after your next, and so on. Therefore there are

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

possible paths! In general, if you have n destinations (not including home) there are

$$n \times (n-1) \times \cdots \times 2 \times 1 = n!$$

paths. That's too many paths for even a computer to handle!

On one hand, we have the greedy algorithm, which is doable but does not always give the best solution. On the other hand, we have the brute force algorithm, which always gives the best solution but is not always doable. Can you think of of a solution that always gives the best solution and is doable? If you can, the world would like to know!

Challenge Problems

1. Ten players participate at a chess tournament. Eleven games have already been played. Prove that there is a player who has played at least three games.

2. Show that the number of people on Earth with an odd number of siblings is even.

3. A round robin baseball tournament has $2n$ participating teams. Two rounds have been played so far. Prove that we can split the teams into two groups of n teams, each so that no teams of the same group have played each other yet.