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Project 2

Geometric Probability and Buffon's Needle Problem

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Content:

2.1. Probability and Random Experiment

Discuss:

Random Experiment

Sample Space

Probability

In this project we'll discuss two different approaches to finding probabilities: empirical and classical (by formula).

2.2. Geometric Probability

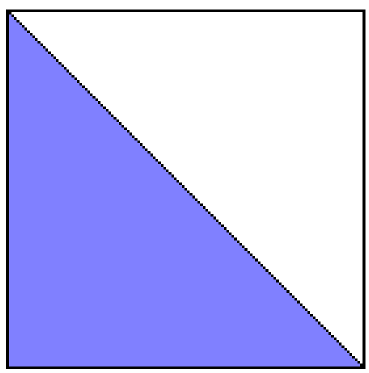
a) Classical approach

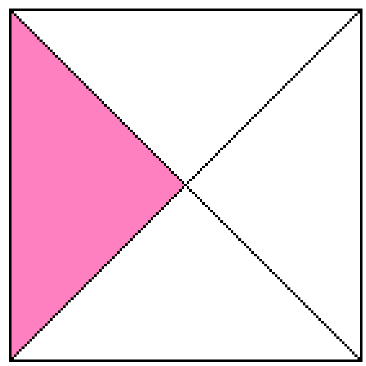
The probability of an event happening is written as P(event).

Assume that a dart is thrown onto the regions below in such a way that the dart is equally likely to hit any point of the region. Find the probability that the dart will hit the shaded region.

The probability will be found by finding the area of the region that is considered a success (the shaded area) divided by the sample space (the total area).

$$\text{P(hitting the target)} = \frac{\text{Area of the target}}{\text{Area of the total space}}$$





a. Let the area of the square = A,

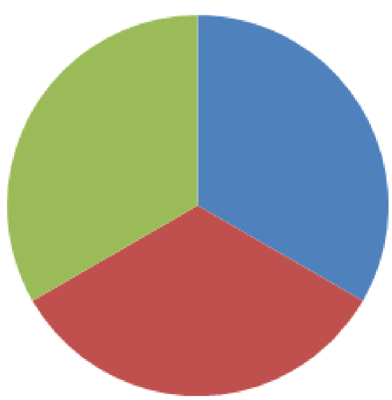
then the shaded area =

P (hitting the shaded region) =

b. Let the area of the square = A,

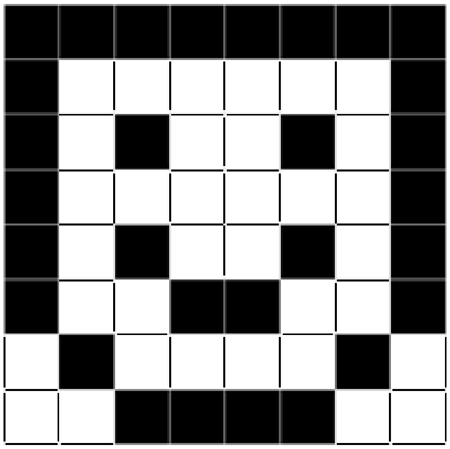
then the shaded area =

P (hitting the shaded region) =



c. Let the area of circle = A, then the area of the red piece =

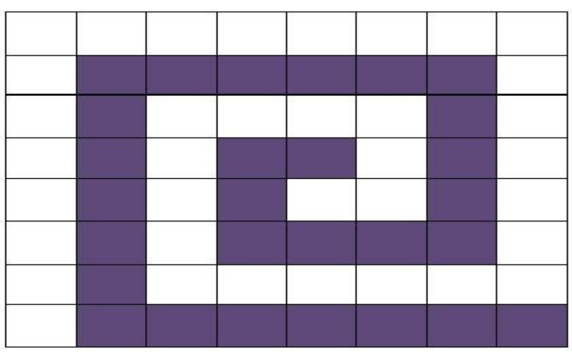
P (hitting the red region) =



d. Let the area of a little square = A,

then the total area of the shaded squares = and the total area of all squares =

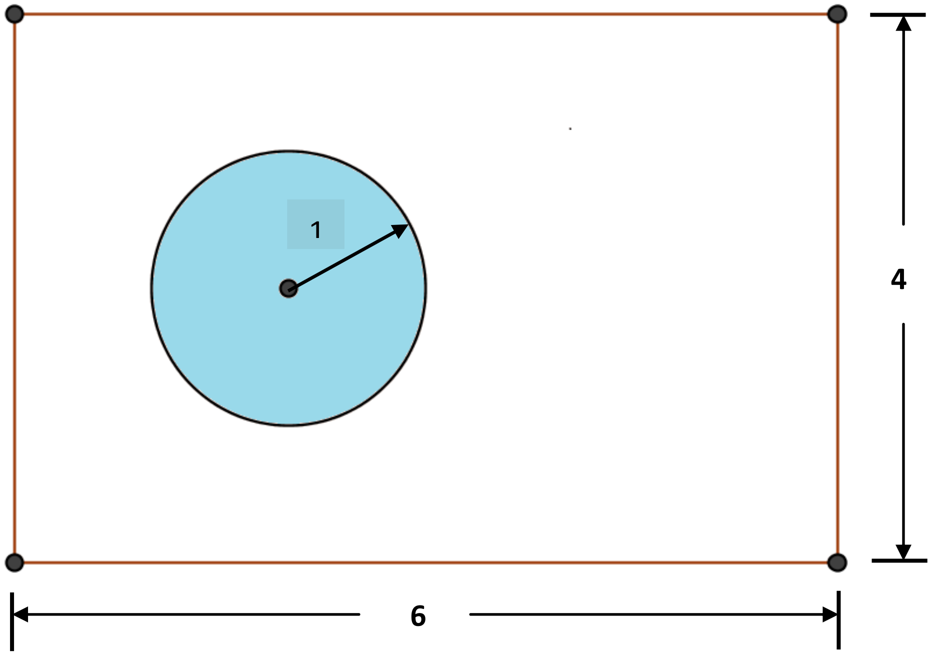
P (hitting the shaded region) =



e. Let the area of a little rectangle = A,

then the total area of the shaded rectangles = and the total area of all rectangles =

P (hitting the shaded region) =



f. What is the probability of hitting the shaded region?

In order to solve this problem, we must know the radius of the circle and the length and the width of the rectangle.

Area of the blue circle =

Area of the rectangle =

$$\text{P(hitting the shaded region)} = \frac{\text{Area of Blue Circle}}{\text{Area of Rectangle}}$$

P (hitting the shaded region) =

2.3 Empirical Approach and Law of Large Numbers

- Repeat an experiment many times under the same conditions to find how often a certain event tends to occur.
- We will find the probability of getting a head on a single coin toss by tossing a coin 20 times and counting how many times it lands heads.

1) The total number of tosses: _____________

2) The number of heads:_______________

3) The proportion of heads in 20 tosses =

- **Law of Large Numbers**: When the total number of tosses is **very large**,

  probability using classical approach ≈ probability using empirical approach.

  Or,

$$\mathrm{P(getting\ a\ head)} \approx \frac{\text{the number of heads}}{\text{total number of tosses}}.$$

Combine all the tosses for your group together to better approximate the probability of getting a head on a single toss.

Total Tosses: ________________________ Number of Heads: _____________________

$\mathrm{P(getting\ a\ head)} \approx$

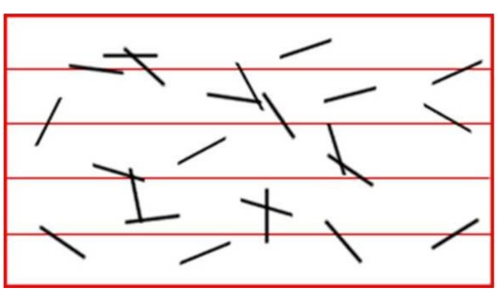
2.4. Buffon's Needle Problem

In 1733, a French nobleman named Georges-Louis Leclerc, the Comte de Buffon, posed the following question:

If a needle of length one unit is dropped at random on a floor covered with boards that are exactly one unit wide, what is the probability that the needle will lie across a line between two strips of wood?

This question became one of the most famous questions in probability and is now known as Buffon's Needle Problem. (Buffon answered his own question in 1777—44 years after he first asked it!)

To answer Buffon's question, we'll first use the empirical approach to find an approximation to the probability. Then we will use the classical approach to find the exact value of the probability.

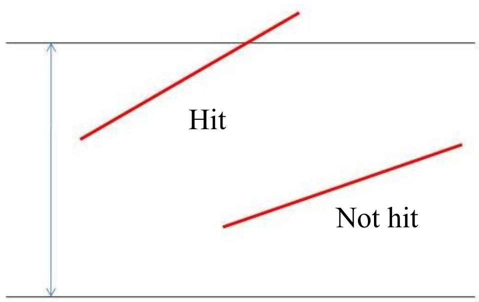


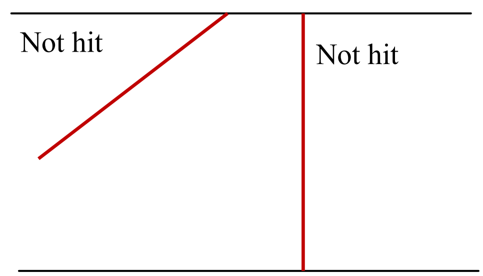
2.4.1. Empirical Approach to Buffon's Needle Problem

Discuss:

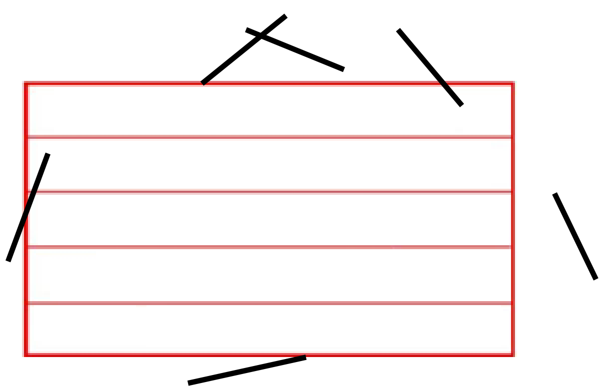
1) What can you do to ensure the needles are randomly dropped?

2) What will count as a "hit"?

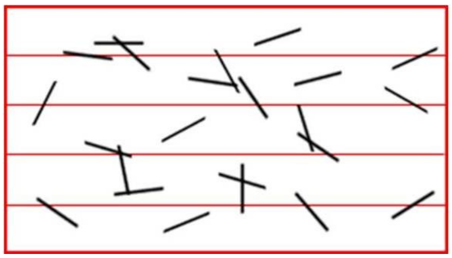




3) What if the needle lands outside the region?



- Perform 150 repetitions of Buffon's Needle experiment and record the results of each experiment.



1) The number of needles dropped ______________________

2) The number of needles hitting the lines________________________

- Approximate the probability of a needle hitting a line by using the Law of Large Numbers

$$\text{P(hitting a line)} \approx \frac{\text{the number of needles hitting a line}}{\text{the total number of needles dropped on the sheet}}$$

$\text{P(hitting a line)} \approx$

Next we will use the classical approach to try to find the theoretical probability of Buffon's Needle Problem.

2.4.2. Classical Approach to Buffon's Needle Problem

First: find the area of the total sample space.

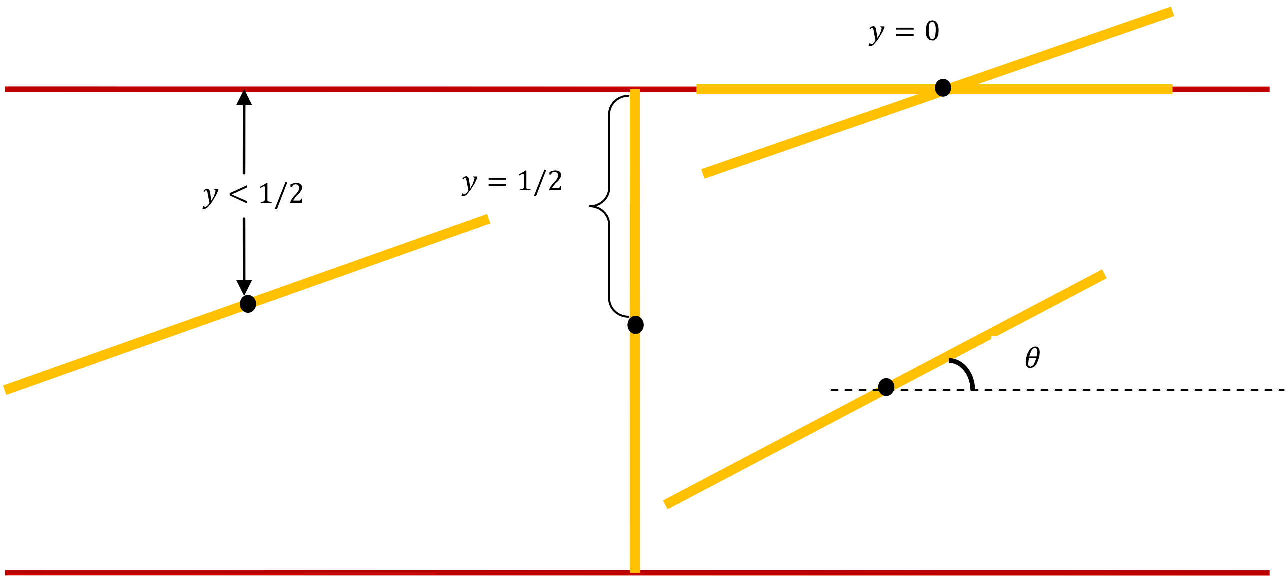
We need to find all the different ways a needle could land on the lined sheet.

1) Find two factors which determine all possible locations of the needle dropped on the lined sheet.

First factor: ____

Second factor: ____

2) For the two factors just found, find the range of the possible values of the factors.



- Let θ be the angle made by the needle and the line

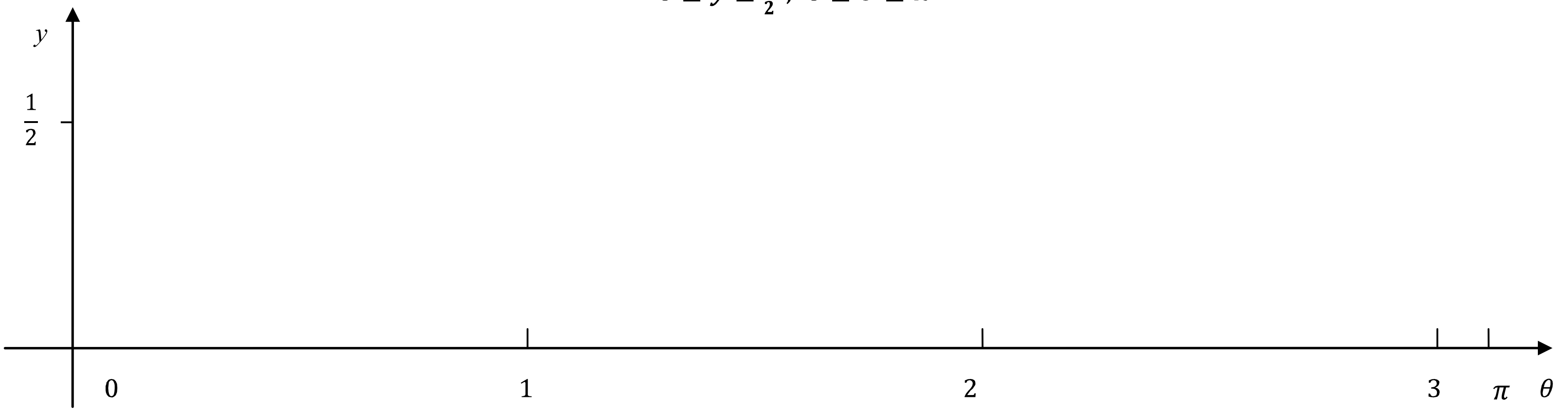
(θ): ______________________________________

- For convenience, let the length of the needle = 1 and let y be the distance from the center of the needle to the nearest line

(y):_______________________________________

3) Plotting one factor along the vertical line (y) and other factor along the horizontal line (θ), find the area (defined by y and θ) that represents ***all possible locations*** of the needles.

$$\mathbf{0 \leq y \leq \tfrac{1}{2}\ ,\ 0 \leq \theta \leq \pi}$$



4) Find the area of the region.

Note: $180^0 = \pi$

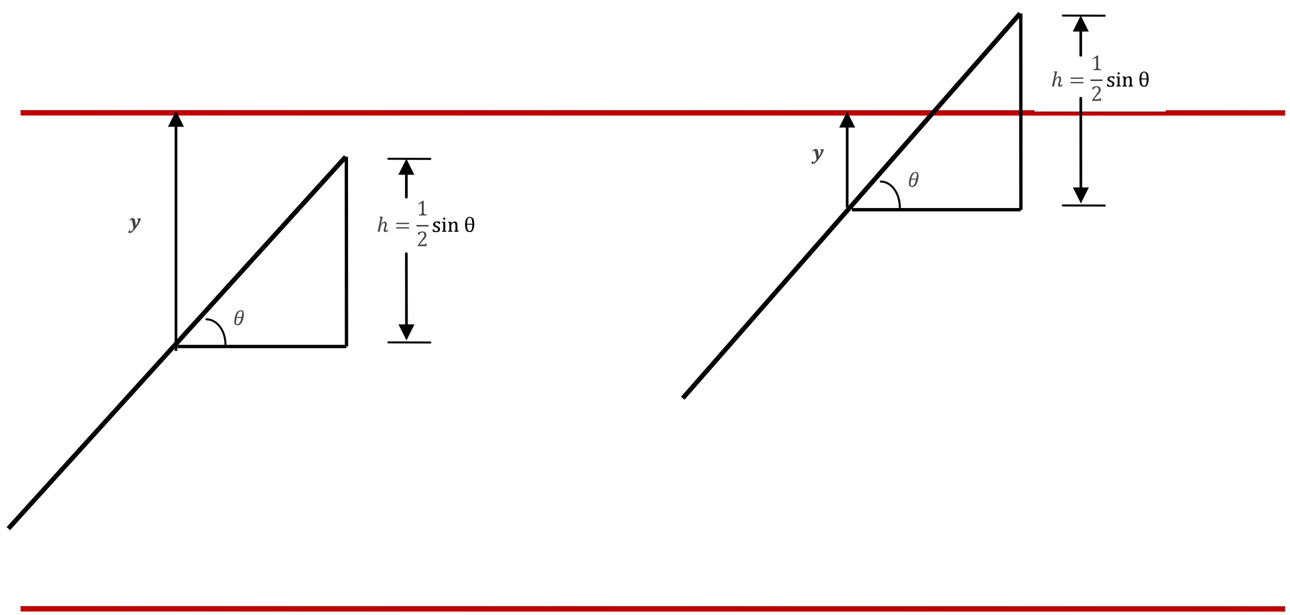
the area of the region =

Goal: Find an area that represents all possible locations of a needle hitting a line and compute the probability.

1) Given a θ, the needle will cross a line if the center of the needle, y, is less than h, where $h = \frac{1}{2}\sin\theta$. Recall that the length of the needle = 1.

   What is the relationship between y and h if the needle does not hit a line?

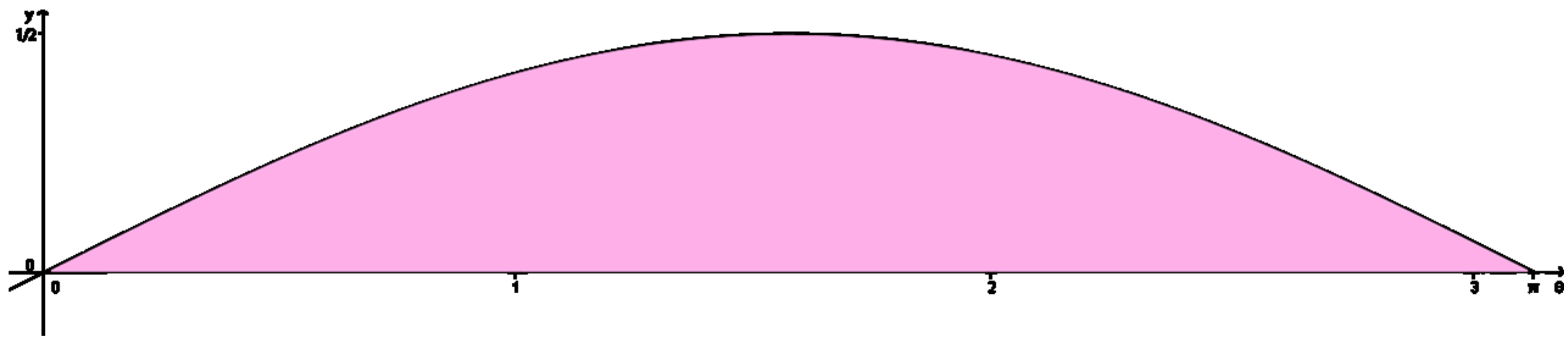
   What is the relationship between y and h if the needle does hit a line?



2) A needle hits a line if y ____________________.

3) Plotting one factor y along the vertical line and factor θ along the horizontal line, the enclosed area below represents all possible locations of the needles that hit a line.

$$\mathbf{0 \leq y \leq \frac{1}{2}\sin\theta\,,\ \ 0 \leq \theta \leq \pi.}$$



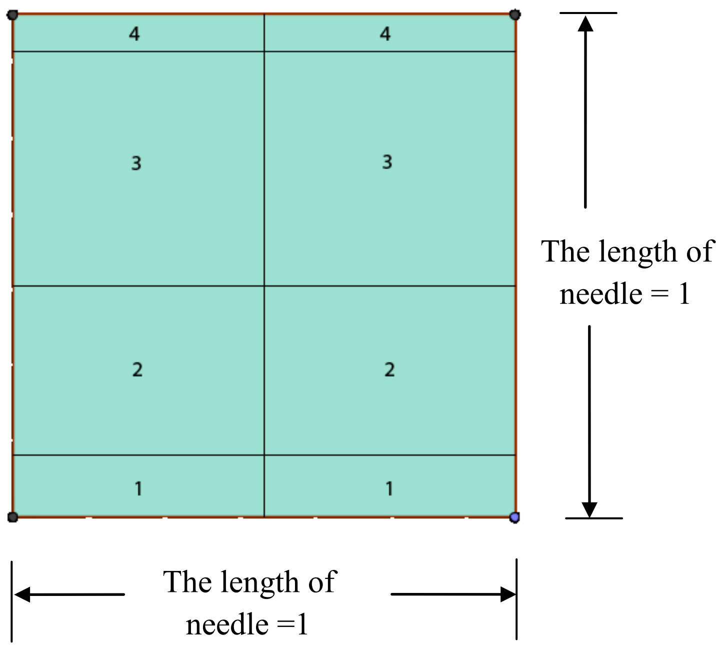
How do we find the area of this region?

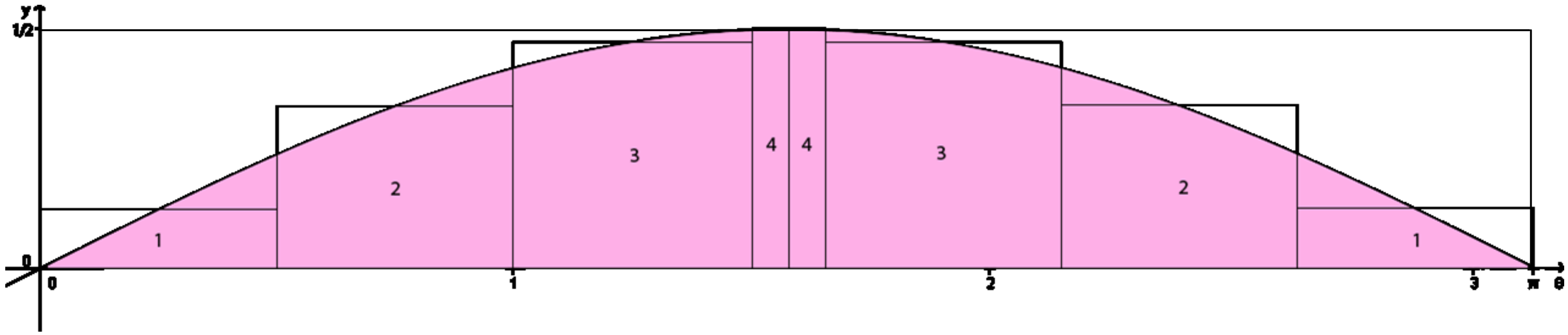
4) Use a Riemann Sum idea to approximate the area in the previous figure.

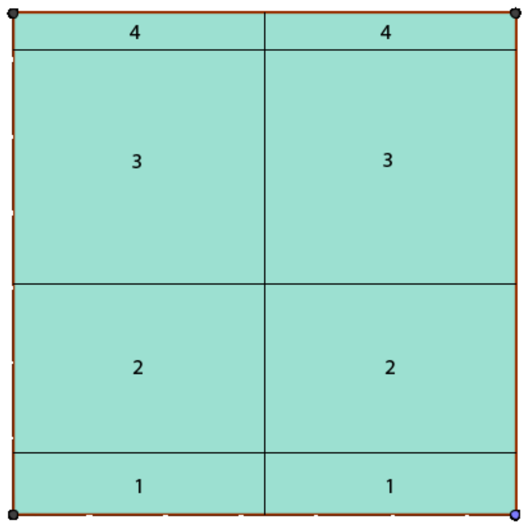
a. What is the area of the large square below? Remember that the length of the needle is 1.

area of the large square below = _________

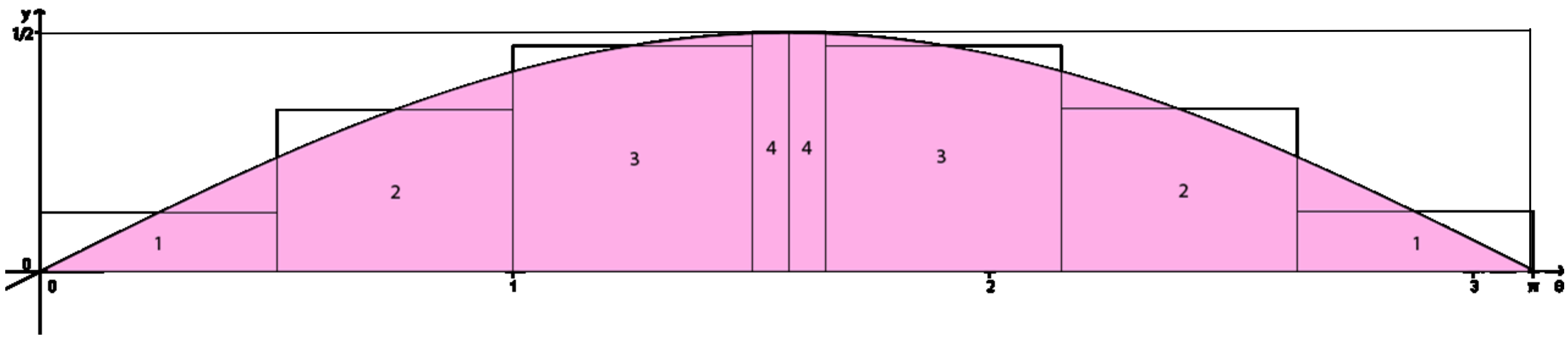
b. Cut the square below into the pieces shown. Then use these pieces to cover the shaded region in the previous figure.




y
1/2
0
1
2
3
4
4
3
2
1
0
1
2
3
π
θ


4
4
3
3
2
2
1
1

c. **The area of the region representing a needle hitting a line** = ________________.



$$0 \le y \le \frac{1}{2}\sin\theta\,, 0 \le \theta \le \pi$$

5) Compute probability of Buffon's Needle Problem using the classical approach.

$$\mathbf{P(needle\ hitting\ a\ line)} = \frac{\textbf{area of shaded region}}{\textbf{area of sample space}}$$

= ________________________________

How close was your empirical approximation to the classical probability?

2.5. Approximating π

Next we will use the probability found in Buffon's Needle Problem to approximate π.

1) Remember that by the Law of Large Numbers

   probability using classical approach ≈ probability using empirical approach

   Then:

$$\frac{2}{\pi} \approx \frac{\text{the number of needles hitting a line}}{\text{total number of needles thrown}}.$$

   Solve for π.

2) The formula to approximate π using Buffon's needles.

$$\boldsymbol{\pi} \approx \frac{2 \times \text{total number of needles thrown}}{\text{the number of needles hitting a line}}$$

3) Take the total number of needles thrown in your experiment and multiply by two. Then divide by the number of needles that hit a line. How close was your approximation of π?

http://xan2-3.deviantart.com/#/d1gnsoa

References:

More information about Buffon's Needle Problem can be found from the following sources:

1. Buffon's Needle Problem, Wolfram MathWorld
http://mathworld.wolfram.com/BuffonsNeedleProblem.html

2. Explanation of solution to Buffon's Needle Problem
http://mste.illinois.edu/reese/buffon/buffon.html

3. Applet simulating random needle dropping

http://www.shodor.org/interactivate/activities/Buffon/

4. (Buffon stamp) http://www.ms.uky.edu/~mai/java/stat/buffon.jpg

5. Wikipedia article http://en.wikipedia.org/wiki/Buffon's_needle

6. http://xan2-3.deviantart.com/#/d1gnsoa