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Art of Counting!

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Some denotes:

1. $\binom{n}{k}$: choose k items from n items
2. $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$.

LEVEL 1

Given a group of people, how many ways can they sit around a table or sit in a line? What if there are some restrictions(like, you want to sit with your best friend!) What if the restrictions become more and more harsh?
1.One person

2.Two persons

3.Three persons

4.Four persons

5.Five persons

If there are n persons here, can you derive the answers for each case?

Is there any connection between these two cases?

LEVEL 2

Now let's do something simpler! Note if you choose 0 person from n persons, then there is only one way.

Choose one from one persons

Choose one from two persons

Choose two from two persons

Choose one from three persons

Choose two from three persons

Choose three from three persons

Choose one from four persons

Choose two from four persons

Choose three from four persons

Choose four from four persons

Choose one from five persons

Choose two from five persons

Choose three from five persons

Choose four from five persons

Choose five from five persons

Is there any connection between ’ n choose k’ and ’n choose n-k’?

Now can you derive the answer to ’choose k persons from n people’?

LEVEL 3

1.How many ways can eight people sit in a circle if Alice and Bob must sit next to each other?

2.How many different mixed-gender committees of 3 people can be chosen from a group of 5 men and 5 women?

3.How many distinct ways can the faces of a cube be painted with 6 different colors? Two paintings are considered identical if the cube can be oriented so that the cubes look the same.

4.How many distinct ways can we arrange four A's and two B's around a circle?

5.Mississippi Rule:the number distinct permutations of MISSISSIPPI.

SOLUTION FOR LEVEL 3

1.Treat Alice and Bob as a single unit and then seat 7 âĂIJpeopleâĂİ in a circle. There are two ways to combine Alice and Bob (Alice on the left or Bob on the left) and 7!/7 = 6! ways to seat 7 in a circle, giving a total of 2*6! = 1440.

2.There are (10 3) was of selecting a committee of three from a group of 5+5=10 people. There are (5 3) ways of choosing an all male committee from the 5 men and (5 3) was of choosing an all female committee from the 5 women. Thus there are (10 3) −2* (5 3) or 120 âĂŞ 2*20 = 80 committees which have at least one man and one woman.

3.Any painting of the cube can be oriented so that a particular color, say red, is on the bottom, so we can assume that we start with a cube that has five blank faces with the bottom painted red. Pick one of the 5 colors remaining to paint the top (5 options), and arrange the 4 remaining colors in a circle and paint the sides accordingly; there are 3! ways to do this. Thus there are a total of 5*3! = 30 ways to paint the cube.

4.If we think about placing n A's and two B's around a circle, we can classify the arrangements by looking at the gaps between the two B's. The B's partition the A's into two runs, and the shortest distance between the B's completely determines the arrangement for we can rotate any arrangement with the same shortest distance line up the B's, and all the other positions are A's so they must match. Thus the number of distinct arrangements is simply the number of possible shortest gap sizes. For four A's the possible shortest gap sizes are 0, 1, and 2 for a total of 3 distinct arrangements.

5.11! / 4!4!2! = 34,650

LEVEL 4

How many regions in space can we get if we cut space by six planes?Here we want no two planes to be parallel and no more than three planes to meet in any single point. In other words, we want to get as many regions as we can.To solve this problem, we would like to begin at simpler point and approach our core problem step by step!

Consider lines cut by point,and then planes cut by lines,finally, space cut by planes! Try to find the answer and fill in the table?

Table 0.1: My caption

n	line by point	plane by line	1space by plane
0			
1			
2			
3			
4			
5			
6			

After you fill in the table (maybe part of the table),have you detect that something else is going on here?

Challenging Problems

1.Central High School is competing against Northern High School in a backgammon match. Each school has three players, and the contest rules require that each player play two games against each of the other school's players. The match takes place in six rounds, with three games played simultaneously in each round. In how many different ways can the match be scheduled?

2.A dessert chef prepares the dessert for every day of a week starting with Sunday. The dessert each day is either cake, pie, ice cream, or pudding. The same dessert may not be served two days in a row. There must be cake on Friday because of a birthday. How many different dessert menus for the week are possible?

ANSWER AND FURTHER DISCUSSION

SOLUTION FOR LEVEL 4

You should be very suspicious that each entry in this table is the sum of the entry directly above and the entry above and to the left. That is exactly the recursion of PascalâĂŹs triangle. If we list of elements of PascalâĂŹs triangle so that they line up according to this recursion, we can see what is happening:

n	line/point	plane/line	space/plane	$k=0$	$=1$	$=2$	$=3$	$=4$	$=5$	$=6$
0	1	1	1	1						
1	2	2	2	1	1					
2	3	4	4	1	2	1				
3	4	7	8	1	3	3	1			
4	5	11	15	1	4	6	4	1		
5	6	16	26	1	5	10	10	5	1	
6	7	22	?	1	6	15	20	15	6	1

Just look at the first five rows about, can you find the feature of each entry in Pascal's triangle?[Remember the work we've done in LEVEL 2].

The number of pieces on a line cut by points is the sum of the first two columns of PascalâĂŹs triangle. For a plane cut by lines, it is the sum of the first three columns. The formula for the number of pieces in space cut by n planes is :

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$$

This leaves several questions that I invite you to explore:

1. Can you make sense of this formula? Given n planes in space, why should the number of regions equal the number of ways choosing none of the planes plus the number of ways of choosing one of the planes plus the number of ways of choosing two of the planes plus the number of ways of choosing three of the planes?
2. What is the formula for the number of finite regions formed by n planes?
3. What happens in higher dimensions, and what does that mean?

SOLUTION FOR CHALLENGING PROBLEMS

1.Let us label the players of the first team A, B, and C, and those of the second team, X, Y, and Z.

1. One way of scheduling all six distinct rounds could be:

- Round 1—->$AX\ BY\ CZ$
- Round 2—->$AX\ BZ\ CY$
- Round 3—->$AY\ BX\ CZ$
- Round 4—->$AY\ BZ\ CX$
- Round 5—->$AZ\ BX\ CY$
- Round 6—->$AZ\ BY\ CX$

The above mentioned schedule ensures that each player of one team plays twice with each player from another team. Now you can generate a completely new schedule by permuting those 6 rounds and that can be done in 6!=720 ways.
2. One can also make the schedule in such a way that two rounds are repeated.
(a)

1. Round 1—->$AX\ BZ\ CY$
2. Round 2—->$AX\ BZ\ CY$
3. Round 3—->$AY\ BX\ CZ$
4. Round 4—->$AY\ BX\ CZ$
5. Round 5—->$AZ\ BY\ CX$
6. Round 6—->$AZ\ BY\ CX$

(b)

- Round 1—->$AX\ BY\ CZ$
- Round 2—->$AX\ BY\ CZ$
- Round 3—->$AY\ BZ\ CX$
- Round 4—->$AY\ BZ\ CX$
- Round 5—->$AZ\ BX\ CY$
- Round 6—->$AZ\ BX\ CY$

As mentioned earlier any permutation of (a) and (b) will also give us a new schedule. For both (a) and (b) the number of permutations are $\frac{6!}{2!2!2!} = 90$
So the total number of schedules is $720 + 90 + 90$ =900.

2. We can count the number of possible foods for each day and then multiply to enumerate the number of combinations.
On Friday, we have one possibility: cake.
On Saturday, we have three possibilities: pie, ice cream, or pudding. This is the end of the week.
On Thursday, we have three possibilities: pie, ice cream, or pudding. We can't have cake because we have to have cake the following day, which is the Friday with the birthday party.
On Wednesday, we have three possibilities: cake, plus the two things that were not eaten on Thursday.
Similarly, on Tuesday, we have three possibilities: the three things that were not eaten on Wednesday.
Likewise on Monday: three possibilities, the three things that were not eaten on Tuesday.
On Sunday, it is tempting to think there are four possibilities, but remember that cake must be served on Friday. This serves to limit the number of foods we can eat on Sunday, with the result being that there are three possibilities: The three things that were not eaten on Monday.
So the number of menus is $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 1 \cdot 3 = 729$.