Math Circles: Pigeons and Rams(ey)

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Sunday, October 2nd, 2016

The "Pigeonhole Principle" is an accepted fact about finite sets, stating that if a collection of N sets (think "pigeonholes") contains a total of $N + 1$ or more elements (think "pigeons"), then some set must contain two or more elements.

A more general version is that if a collection of N sets contains a total of $KN + 1$ or more elements, then some set must contain at least $K + 1$ elements.

The questions below explore some of the consequences of this idea. When asked if something must happen, either give a reason why it must or else show an example where it fails to happen.

1. Suppose there are 2 pigeonholes and 3 pigeons.
   - Must some pigeonhole contain two or more pigeons?
   - Must some pigeonhole contain exactly two pigeons?
2. Suppose there are 2 pigeonholes and 4 pigeons.
   - Must some pigeonhole contain three or more pigeons?
   - Must both pigeonholes contain two or more pigeons?
   - Must each pigeonhole contain at least one pigeon?
3. Suppose there are 10 (huge) pigeonholes and 200 pigeons.
   - Must some pigeonhole contain 20 or more pigeons?
   - Can you think of an arrangement in which no pigeonhole contains exactly 20 pigeons?
4. Over a million Christmas trees were sold in California last winter. No tree has more than 800,000 needles on it. Show that two of the Christmas trees had the same number of needles on them at midnight of the night before Christmas.

5. A bag contains 10 black marbles and and 10 white marbles. For some reason you can't tell them apart.
   - What is the smallest number of marbles that you must pull out to guarantee that you get at least two marbles of the same color?
   - What is the smallest number of marbles that you must pull out to guarantee that you get at least two marbles of DIFFERENT colors?
6. Imagine that you own 7 pairs of socks, each pair is a different color, and all 14 socks are loose in the dryer. You are doing the laundry blindfolded.
   - How many socks will you have to pull out to guarantee that you get at least two of the same color?
   - How many socks will you have to pull out to guarantee that you get at least two of DIFFERENT colors?
7. What is wrong with the following version of the pigeonhole principle: If there are more pigeonholes than pigeons, then at least one pigeon must be in two pigeonholes at once.
8. How many people must you have before you can be certain that two of them share the same first initial?
9. Prove that in any set of 11 consecutive counting numbers, there must be two that have the same ones digit.
10. Prove that in any set of 11 consecutive counting numbers, there must be two whose difference is evenly divisible by 10.

11. Prove that in any set of $N + 1$ consecutive counting numbers, there must be two whose difference is evenly divisible by N. (Hint: use base N arithmetic.)

12. Use the pigeonhole principle to remove the "consecutive" condition: prove that in any set of $N + 1$ counting numbers there must be two whose difference is evenly divisible by N.

13. A "power of three" is an integer of the form 3^n, where $n = 0, 1, 2, \ldots$. The smallest powers of three are 1,3,9,81,243,....

    - Prove that there exist two powers of three which differ by a multiple of 2016.

    - Prove that there exist 2016 powers of three each pair of which differs by a multiple of 2016.

    - Prove that, given any positive integers A and B, there exist A powers of three each pair of which differs by a multiple of B.

Ramsey's Theorem is an application of the pigeonhole principle to subgraphs of a graph rather than elements of sets.

14. Draw a triangle and label its vertices with any three numbers. Label each edge with the difference between the numbers at its endpoints, then color the edge ODD if the difference is odd and EVEN if the difference is even.

    - Is one of your edges colored EVEN?
    - Prove that no matter which three numbers were chosen, at least one edge must be colored EVEN.

15. Draw a square and label its vertices with any four numbers. Also draw the two diagonal edges connecting he corners. Label each edge with the difference between the numbers at its endpoints, then color the edge ODD if the difference is odd and EVEN if the difference is even.

    - Is one of your edges colored EVEN?
    - Prove that no matter which four numbers were chosen, at least one edge must be colored EVEN.

16. A *complete graph on N vertices* is a figure consisting of N points (called *vertices*) with every pair connected by a line segment (called an *edge*). For example, a complete graph on 3 points is a triangle, and a complete graph on 4 points is a square with both diagonals.

    - Draw a complete graph on 5 vertices.

    - How many edges are there in a complete graph on 3 vertices? 4 vertices? 5 vertices?
    - Prove that the number of edges in a complete graph on N vertices is $N(N-1)/2$. (Check that this agrees with your results for $N = 3, 4, 5$.)

17. *Two-coloring* a complete graph is to color each of its edges one of two colors. We may ask, must there be a triangle all in one color?

    - Two-color the complete graph on 3 vertices so that there is NO triangle all of one color.

    - Two-color the complete graph on 4 vertices so that there is NO triangle all of one color.

    - Two-color the complete graph on 5 vertices so that there is NO triangle all of one color.

Notice how, as the number of vertices increases, it is harder to avoid the one-color triangle.

18. One generalization of the pigeonhole principle states that any two-coloring of a complete graph on sufficiently many vertices must contain a one-color triangle.

    - Draw a few complete graphs on 6 vertices.

    - Try to two-color the graphs so that there is NO one-color triangle. (warning: you may not succeed.)

19. See if you can prove the conjecture: In any two-coloring of a complete graph on 6 vertices, there must be a triangle all of one color.

20. *Ramsey's theorem* is a precise statement of this generalization of the pigeonhole principle: For every pair of positive integers P, Q, there is a least positive integer $R = R(P, Q)$ such that any two-coloring of a complete graph on R or more vertices must contain a one-color complete graph on P vertices or else a one-color complete graph on Q vertices. Explain how the conjecture above is equivalent to the claim that $R(3, 3) \leq 6$.

21. Given that $R(3, 3) \leq 6$, prove that $R(3, 3) = 6$ by referring to your two-colored complete graph on 5 vertices.

Challenge problems:

C1: Ramsey's theorem may be proved by induction using the formula

$$R(r,s) \leq R(r1,s) + R(r,s1), \qquad r \geq 1,\ s \geq 1.$$

- See if you can prove this formula.

- See if you can prove Ramsey's theorem from this formula.

C2: State a generalization of Ramsey's theorem to the case where there are $c > 2$ colors and we are concerned with one-color complete subgraphs on $n_1, \ldots, n_c$ points.

C3: The generalized Ramsey theorem on c colors may be proved by induction using the formula

$$R(n_1, \ldots, n_c) \leq R(n_1, \ldots, n_{c-2}, R(n_{c-1}, n_c)) \qquad n_1, \ldots, n_c \geq 1.$$

- See if you can prove this formula.

- See if you can prove the generalized Ramsey theorem from this formula.