Cardinality of Sets

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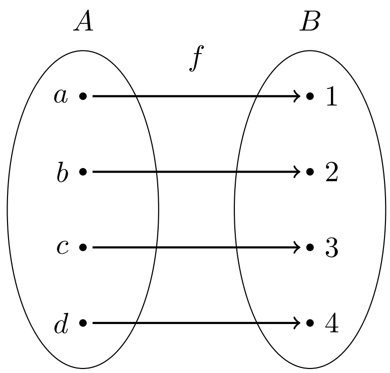
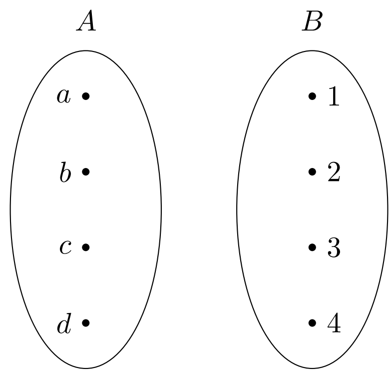
The cardinality of a finite set A is just the number of elements of A, denoted by $|A|$. For example, $A = \{a, b, c, d\}$, $B = \{n \in \mathbb{Z} : -3 \leq n \leq 3\} = \{-3, -2, -1, 0, 1, 2, 3\}$. Then we have $|A| = 4 < |B| = 7$. The cardinality of an infinite set is trickier. We will meet both finite and infinite sets below, but the main point is explaining how there are numerous different kinds of infinity, and some infinities are bigger than others.

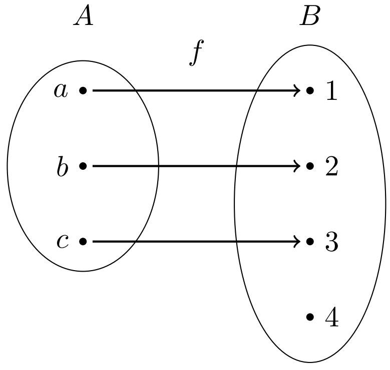
Definition 1 Two sets A and B have the **same cardinality**, denoted by $|A| = |B|$, if there exists a **one-to-one correspondence** $f : A \to B$, where each element of A is paired with exactly one element of B, and each element of A is paired with exactly one element of A. It is also called as **bijective function** or **bijection**.

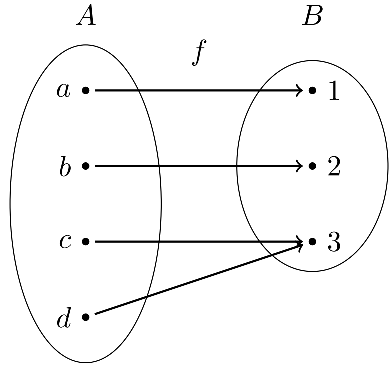
Example 1 The picture on the left gives a bijection between two sets A and B. Can you write down another bijection between the two sets on the right?





Example 2 If A and B are as indicated in either of the following figures, then there can be no bijection $f : A \to B$.





Exercise 1 The table below shows a bijection between natural numbers $\mathbb{N}$ and integers $\mathbb{Z}$. Even though $\mathbb{Z}$ seems roughly twice as large as $\mathbb{N}$, because $\mathbb{Z}$ has all the negative integers as well as the positive ones and zero.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$f(n)$	0	1	-1	2	-2	3	-3	4	-4	5	-5	6	-6	$\cdots$

(1) Find $f(17)$ and $f(24)$, and describe the bijection above as an explicit formula. (Hint: treat even n's and odd n's separately)
(2) Find f^{-1}, the inverse function of f. In other words, given an arbitrary integer, find the corresponding natural number in the bijection above.

Exercise 2 Find a bijection between $\mathbb{N}$ and $\{0, 1\} \times \mathbb{N}$. $\{0, 1\} \times \mathbb{N}$ is the set of duples $(0, n)$ and $(1, n)$, where n is a natural number. For example, (0,1), (0,3), (1,5) $\cdots$

Definition 2 Given a set A, then A is **countably infinite** if $|\mathbb{N}| = |A|$, that is, if there exists a bijection f between natural numbers and the elements of A. In other words, the elements of A can be enumerated in an infinite list $a_1, a_2, a_3, a_4 \cdots$. Countable infinity is the smallest infinity. A is **uncountable** if A is infinite and $|\mathbb{N}| < |A|$, so there exists *no* such bijections.

Example 3 Show the set $\mathbb{Q}$ of rational numbers is countably finite. A rational number has the form $\frac{a}{b}$, where a, b are integers and $b \neq 0$.

To prove this, we need to write the rational numbers in a list form. This can be done by the following table. The top row has a list of all integers. Each column headed by an integer k contains alll the fractions (in reduced form) with numerator k. For example, the column headed by 2 contains the fractions $\frac{2}{1}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7} \cdots$, but not $\frac{2}{2}, \frac{2}{4}, \frac{2}{6} \cdots$ because those are not reduced. You should examine the table and convince yourself that it contains all rational numbers.

0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now check the bottom-left to top-right diagonals. For the first diagonal, you get $\frac{0}{1}$; for the second diagonal, you get $\frac{1}{1}$; for the third and fourth diagonals, you get $\{\frac{1}{2}, \frac{-1}{1}\}$ and $\{\frac{1}{3}, \frac{-1}{2}, \frac{2}{1}\}$ respectively$\cdots$ Continue counting on the diagonals, we get an infinite list of all rational numbers. Write down the first 16 elements of the list.

Exercise 3 (1) If A and B are countably infinite, show that $A+B$ is also countably infinite. $A+B$ contains all the elements of A and B.
(2) Show that $\mathbb{N} \times \mathbb{N}$ is countably infinite. $\mathbb{N} \times \mathbb{N}$ is the set of duples (a, b), where both a and b are natural numbers. For example, $(1, 2)$, $(3, 5)$, $(4, 3) \cdots$ (Hint: it is similar to the Example 3)
In fact, this can be easily extended to saying if A and B are countable, then $A \times B$ is also countable.

Exercise 4 (1) Partition $\mathbb{N}$ into 6 infinite subsets. In other words, find 6 infinite sets $A_1, A_2, \cdots, A_6$, such that none of the two sets have a common element, and $A_1 + A_2 + \cdots + A_6 = \mathbb{N}$, the natural numbers.
(2) Partition $\mathbb{N}$ into infinitely many infinite subsets.

Exercise 5 2^A is the power set of A, it contains all the subsets of A (so 2^A is the set of sets!). For example, $2^{\{1,2\}}$ contains 4 elements: the empty set $\emptyset$, $\{1\}$, $\{2\}$ and $\{1,2\}$. 2^A always contain the empty set $\emptyset$ and A itself.
(1) Write down all the elements of $2^{\{a,b,c\}}$, and write down any 4 elements of $2^{\mathbb{N}}$.

(2) Show that there exists a bijection between $2^{\mathbb{N}}$ and the set T of all infinite sequences of binary digits. An infinite sequence of binary digits looks like $(0,1,1,0,1,0,0,1\cdots)$

(3)(Cantor's diagonal argument) Prove that T and $2^{\mathbb{N}}$ are uncountable. Consider an infinite sequence of T: $s_1, s_2, \cdots, s_n, \cdots$, for example the one below:

$$\begin{aligned}
s1 &= (\mathbf{0}, 0, 0, 0, 0, 0, 0, ...)\\
s2 &= (1, \mathbf{1}, 1, 1, 1, 1, 1, ...)\\
s3 &= (0, 1, \mathbf{1}, 1, 0, 1, 0, ...)\\
s4 &= (1, 0, 1, \mathbf{0}, 1, 0, 1, ...)\\
s5 &= (1, 1, 0, 1, \mathbf{0}, 1, 1, ...)\\
s6 &= (0, 0, 1, 1, 0, \mathbf{1}, 1, ...)\\
s7 &= (1, 0, 0, 0, 1, 0, \mathbf{1}, ...)\\
&\vdots
\end{aligned}$$

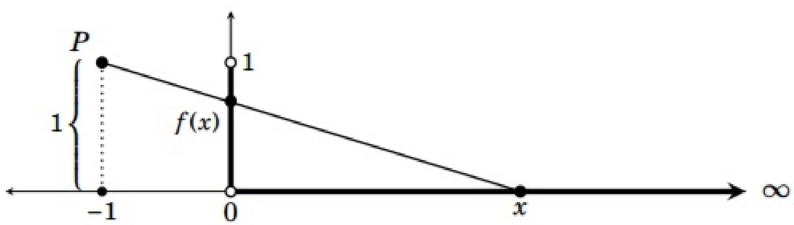
We highlighted the diagonal of the sequence, which is $(0,1,1,0,0,1,1,\cdots)$. Now find an element of T that cannot appear in the sequence above and give the reason. What do we need to change if we replace the sequence above by another sequence like the one on the next page? Conclude that T and $2^{\mathbb{N}}$ are uncountable from our definition.

$$
\begin{aligned}
s1 &= (1,1,0,0,0,1,0,...)\\
s2 &= (1,0,1,0,1,0,1,...)\\
s3 &= (1,1,1,0,0,1,1,...)\\
s4 &= (0,0,0,0,1,1,1,...)\\
s5 &= (1,1,0,1,0,0,0,...)\\
s6 &= (1,0,0,1,0,0,1,...)\\
s7 &= (1,1,0,1,1,0,1,...)\\
&\vdots
\end{aligned}
$$

Exercise 6 (1) A real number r can be written in the decimal representation $r = a_0.a_1a_2a_3\cdots$, where a_0 is an integer and $a_1, a_2, a_3, \cdots$ are integers satisfying $0 \leq a_n \leq 9$, n goes from 1 to infinity. For example $0.10000\cdots$ and $123.45678\cdots$ are real numbers. Use the results from Exercise 4 to prove that the set $\mathbb{R}$ of real numbers is uncountable.
(2) Prove that the set of irrational numbers is uncountable. An irrational number is a real number but not a rational number.

Exercise 7 We try to find a bijection between $(0,\infty)$ and $(0,1)$, where $(0,\infty)$ means the line segment from 0 to infinity (0 not included), and $(0,1)$ means the line segment from 0 to 1 (0 and 1 not included).

Consider the interval $(0,\infty)$ as the positive x-axis below, and $(0,1)$ be on the y-axis. So that $(0,\infty)$ and $(0,1)$ are perpendicular to each other.
The figure also shows a point $P=(-1,1)$ (This means the coordinate of a point in a 2-dimensional space, not a line segment as above). Define $f(x)$ to be the point on the y-axis where the line from P to x intersects the y-axis.

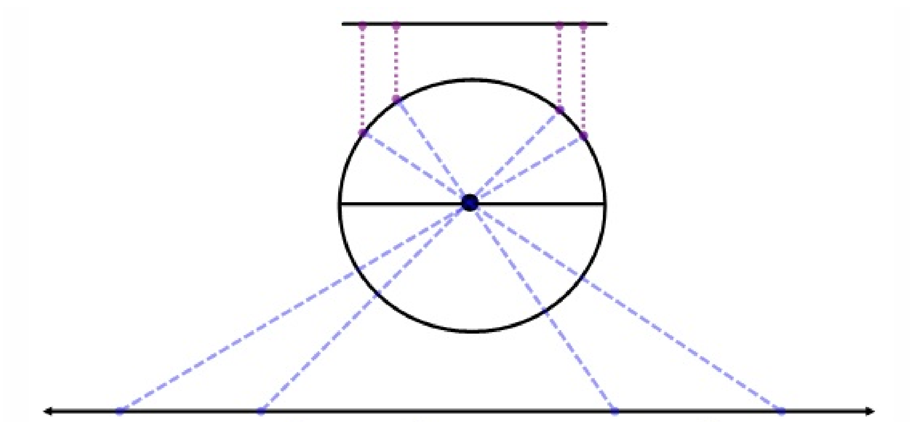


(1) Find $f(1), f(3)$, and the explicit formula of $f(x)$. (Hint: use congruence of triangles)

(2) If we replace $P=(-1,1)$ by $(-2,1)$, do we still get a bijection between $(0,\infty)$ and $(0,1)$? If so, write down the formula. What if we replace $P=(-1,1)$ by $(-1,2)$?

Exercise 8 (1) Prove that the rational numbers that are greater than 0 but less than 1 are countably infinite.
(2) Use (1) to find a bijection between $[0, 1]$ and $(0, 1)$, where $[0, 1]$ means the line segment from 0 to 1 (0 and 1 included). (Hint: the bijection cannot be continuous. Consider rational numbers and irrational numbers separately.)

Exercise 9 The picture below illustrates a bijection from $(0, 1)$ to $\mathbb{R}$.



Find some other bijections between $(0, 1)$ and $\mathbb{R}$. The easiest way will be using trigonometric functions. Another way is to find a bijection between $\mathbb{R}$ and $(0, \infty)$, and combine it with the bijection between $(0, \infty)$ and $(0, 1)$.

Exercise 10 (Challenging) We learned power sets from Exercise 4. Now prove that $|A| < |2^A|$, in other words, the cardinality of A is always less than the cardinality of its power set for any A, not just countable ones.

The statement is obvious if A is finite. If A is infinite, it suffices to show that there cannot be a bijection between A and 2^A. Assume on the contrary that such a bijection $f : A \to 2^A$ exists, we want to show some contradiction. Notice that for any element x in A, $f(x)$ is a subset of A. Hence either x is an element of the subset $f(x)$ or not. Consider the set $B = \{x \in A : x \notin f(x)\}$, which contains every x that is not an element of $f(x)$.

Now B is also a subset of A, hence an element of 2^A. Since $f : A \to 2^A$ is a bijection, there exists some a in A such that $f(a) = B$. Derive a contradiction from our reasoning above.