Relations Between Problems

It is common that the same math problem appears in many different forms. It's helpful to be able to recognize when this happens. If two problems are equivalent, then you only need to solve one of them! Moreover, sometimes it is easier to transform the original problem in such a way that the solution doesn't change, but the solution is easier to find. First, I give a fun (but very advanced!) historical example which is still relevant for modern mathematics. Read this at home if you're interested! For now, skip to page 3.

(Advanced) The Riemann Hypothesis

In 1859, Bernhard Riemann made the following conjecture.[1] The Riemann zeta function $\xi(s)$ is defined in the half-plane $\mathcal{R}(s) > 1$ by the convergent Dirichlet series

$$\xi(s) = \sum_{n=1}^{\infty} n^{-s},$$

and is extended by analytic continuation to the complex plane, where it has one singularity: a simple pole with residue 1 at $s = 1$. The Riemann hypothesis states that the nonreal zeros of the Riemann zeta function $\xi(s)$ all lie on the line $\mathcal{R}(s) = 1/2$.

It turns out that the Riemann hypothesis is connected to the distribution of prime numbers, and this partially explains why mathematicians find it so interesting. It is regarded as one of the most important unresolved problems in mathematics, and consequently is one of the Clay Mathematics Institute's Millennium Prize Problems. Throughout history, one approach to proving or disproving the Riemann hypothesis has been to study other problems which are closely related, for which a different set of tools may be helpful.

There are many problems which are equivalent to the Riemann hypothesis, which can be found in books by Edwards[2] and Titchmarsh.[3] In 1982, Robin (using results of Ramanujan from 1915) proved that, *if the Riemann hypothesis is true*, then

$$\sigma(n) < e^{\gamma} n \log\log n$$

holds for sufficiently large n, where $\gamma \approx 0.57721$ is Euler's constant, and $\sigma(n)$ is the *sum-of-divisors function*. That is, $\sigma(n)$ is computed as the sum of all (unique) divisors of n. For example,

$$\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28, \quad \sigma(6) = 1 + 2 + 3 + 6 = 12.$$

[1] Here we follow the presentation of Jeffrey C. Lagarias (2002). An elementary problem equivalent to the Riemann hypothesis. *American Mathematical Monthly* **109**(6), 534–543.

[2] Chapter 12 of Edwards, H.E. (1974). *Riemann's Zeta Function.* Academic Press: New York.

[3] Chapter 14 of Titchmarsh, E.C. (1986), revised by D.R. Heath-Brown. *The Theory of the Riemann Zeta Function*, Second Edition. Clarendon Press: Oxford.

Using this result, Lagarias (2002) showed that the following problem is equivalent to the Riemann hypothesis.

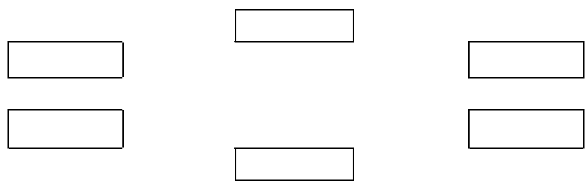
Let $H_n = \sum_{j=1}^{n} 1/j$. Show that, for each $n \geq 1$,

$$\sum_{d|n} d \leq H_n + \exp(H_n)\log(H_n),$$

with equality only for $n = 1$. The function $\sigma(n) = \sum_{d|n} d$ is the *sum-of-divisors function*, as defined above. The number H_n is called the nth *harmonic number*, which is the sum of the reciprocals of the first n natural numbers. Though this type of equivalent problem has not yet yielded a resolution of the Riemann hypothesis, it is believed that such equivalent problems are quite useful to furthering our understanding of the nature of the problem, and can yield insights which will ultimately assist in resolving the original problem.

Relations Between Problems, I

- *Below are seven problems, labeled A,B,C,D,E,F,G. The object is to place the letter of each problem in one of the boxes below.*
- *Put letters in the same box if they are mathematically the "same" problem, apart from superficial differences of context.*
- *If problems in different boxes are closely related mathematically, connect their boxes by a line, or by a double line if the connection is very strong. (Note, you need not use all of the boxes, and you may reasonably answer this question even if you have not completely solved the individual problems.)*
- *Work on this individually for a few minutes. Then compare answers in your group.*
- *Try to come to some consensus on how to* ***explain*** *(to the whole group) your choices, in particular the nature of the connections.*



A. What are all three-digit numbers that you can make using each of the digits 1, 2, 3, and using each digit only once?
B. You row around a small island on which there are a tree, a cabin, and a flag pole. As you look at the island, you see them in some order, left, middle, right. What are all the different orders in which you see them as you row completely around the island?*
C. In a group of nine students, how many ways are there to pick a team of three students?
D. If Angel, Barbara, and Clara have a race, and there are no ties, what are all possible outcomes: first, second, third?
E. From a bag full of many pennies, nickels, and dimes, I randomly choose three coins. What are all possible amounts of money that I might have?
F. In a 3 x 3 grid square, color some of the nine (unit) squares blue, in such a way that there is exactly one blue square in each row and in each column. What are all ways of doing this?
G. What are all the symmetries of an equilateral triangle?

* Not in a straight line

The first two activities were created by Hyman Bass, Professor, University of Michigan; http://www.math.lsa.umich.edu/WCMTC/past-events.html

Relations Between Problems, II

- *Below are seven problems, labeled A,B,C,D,E,F,G. The object is to place the letter of each problem in one of the boxes below.*
- *Put letters in the same box if they are mathematically the "same" problem, apart from superficial differences of context.*
- *If problems in different boxes are closely related mathematically, connect their boxes by a line, or by a double line if the connection is very strong. (Note, you need not use all of the boxes, and you may reasonably answer this question even if you have not completely solved the individual problems.)*
- *Work on this individually for a few minutes. Then compare answers in your group.*
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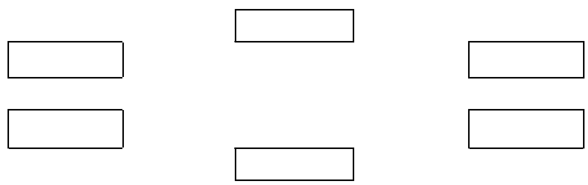

A. A taxi wants to drive (efficiently) from one corner to another that is 5 blocks north, and 3 blocks east. How many possible routes are there to do this?

B. On the number line, starting at 0, you are to take 8 steps, each of which is either distance 1 to the right, or distance 1 to the left, and in such a way that you end up at -2. How many different such walks are there?

C. The home team won a soccer game 5 to 3. How many possible sequences of scoring were there as the game progressed?

D. You have coins worth 3¢ and 5¢. With 8 such coins, how many different values can you obtain?

E. From a group of 8 students, you need to select a (5-person) basketball team. How many different ways are there to do this?

F. You are to cut a 9-inch ribbon into six pieces, each of length a whole number of inches. How many ways are there to do this?

G. In the expansion of $(1 + x)^8$, what is the coefficient of x^3?

Explain why the original problem is equivalent to the related problem.

Problems

1. **Original Problem:** In how many ways can you choose 2 oranges out of 7?
   - *Related Problem:* In how many ways can you choose 5 apples out of 7?
2. **Original Problem:** Suppose you have 10 blocks of different sizes, and each block is a cube. The lengths in centimeters of the sides of the 10 blocks are $\{1, 2, 3, \ldots, 10\}$. Using all of the blocks, can you build two block towers of the same height by stacking cubes on top of one another?
   - *Related Problem:* Is it possible to divide the numbers $\{1, 2, \ldots, 10\}$ into two sets whose sums are equal?
3. **Original Problem:** A circle of radius 5 intersects another circle, radius 10, at right angles. What is the difference of areas of the non-overlapping portions?[4] *Hint: Some information here may be a red herring.*
   - *Related Problem:* Let A and B be the areas of the two non-overlapping pieces such that the area of one shape (including the overlapping portion) is $A + X$ and the other area (including the overlapping portion) is $B + X$. What is the difference of areas of the non-overlapping portions?
4. **Original Problem:** Find the value of a which maximizes
$$\prod_{i=1}^{n}(y_i - ax_i)^2,$$
   where $(y_1, \ldots, y_n)$ and $(x_1, \ldots, x_n)$ are (scalar) real numbers.
   - *Related Problem:* Find the value of a which minimizes
$$-\log\left(\prod_{i=1}^{n}(y_i - ax_i)^2\right),$$
     where log means the natural logarithm (or your favorite base).
5. **(Harder) Original Problem:** Suppose you have a line segment $[0, 1]$. You choose two points on this segment *at random* (in the sense that all values on the segment are equally likely to be chosen). These two points divide the segment into three smaller segments. What is the probability that the three smaller segments can be the sides of a triangle?

[4]This and many other 'generalizations' of simpler problems can be seen here: `http://www.cut-the-knot.org/Generalization/epairs.shtml`

- *Related Problem:* What is the probability that none of the three smaller segments exceeds 1/2 in length?

6. **(Harder) Original Problem:** Given 12 integers, show that 2 of these can be selected such that their difference is divisible by 11.

   - *Related Problem:* Let $\{a_1, \ldots, a_{12}\}$ be the set of remainders of these integers mod 11. Show that at least 2 of these numbers are equal.