Counting Methods for Calculating Probabilities

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0. Independence

Two things are independent if the chances for the second given the first are the same, no matter how the first one turns out. Otherwise, the two things are dependent.

Example: Someone is going to toss a coin twice. If the coin lands heads on the second toss, you win a dollar.
a) If the first toss is heads, what is your chance of winning the dollar?
b) If the first toss is tails, what is your chance of winning the dollar?

Solution: If the first toss is heads, there is a 50% chance to get heads the second time. If the first toss is tails, the chance is still 50%. The chances for the second toss stay the same, however the first toss turns out. Each toss is independent.

Exercises:
1) A fair coin is tossed 6 times. If the first 5 tosses are tails, find the chance that the 6^{th} toss is a head.

2) A fair six-sided die is rolled 4 times. If the first 3 rolls are all smaller than 3, find the chance that the 4^{th} roll is greater than or equal to 3. (Note: Each roll is independent.)

1. Multiplication Rule

The chance that two things will both happen equals the chance that the first will happen, multiplied by the chance that the second will happen given the first has happened. Similarly, the chance that three things will all happen equals the chance that the first will happen, multiplied by the chance that the second will happen given the first has happened, and then multiplied by the chance that the third will happen given the first two have already happened. The conclusion can be generalized to the chance that *n* things will all happen.

Example: A deck of cards without two jokers is shuffled, and two cards are dealt. What is the chance that both are aces?

Solution: The chance that the first card is an ace equals 4/52. Given that the first card is an ace, there are 3 aces among the 51 remaining cards. So the chance of a second ace equals 3/51. The chance that both cards are aces equals $(4/52)\times(3/51) \approx 0.452\%$.

Exercises:
1) A deck is shuffled and two cards are dealt. Find the chance that the first card is a heart and the second card is also a heart.

2) A die is rolled three times. Find the chance that the first roll is 1, the second roll is 2, and the third roll is 3.

3) A die is rolled three times.
a) Find the chance that all the rolls show 3 or more spots.

b) Find the chance that none of the rolls show 3 or more spots.

c) Find the chance that not all the rolls show 3 or more spots.

4) A deck is shuffled and four cards are dealt.
a) Find the chance that all four cards are aces.

b) Find the chance that at least one card is not an ace.

c) Find the chance that none of the four cards is an ace.

d) Find the chance that at least one card is an ace.

5*) (Birthday Problem) Suppose that there are 30 people in the classroom. Find the chance that at least two have the same birthday. (For simplicity, assume that there are 365 days in a year. Leap year will not be considered.)

Note: The correct answer is around 70.6%. This chance would increase to 89.1% if there are 40 people in the classroom, and 97.0% for only 50 people.

2. Addition Rule

The chance that at least one of two things will happen equals the chance that the first thing will happen plus the chance that the second thing will happen, and then minus the chance that both things will happen.

Example: A card is dealt off the top of a well-shuffled deck. What is the chance for it to be in a major suit (hearts or spades)?

Solution: Let us use addition rule to solve this problem. The question asks for the chance that at least one of the following two things will happen: (i) the card is a heart; (ii) the card is a spade. The chance that (i) will happen is 13/52 since there 13 hearts in the deck. The chance that (ii) will happen is also 13/52. The chance that both (i) and (ii) will happen equals zero because if (i) happens, (ii) cannot happen, and if (ii) happens, (i) cannot happen either. So the chance of getting a card in a major suit is 13/52+13/52-0=1/2 by addition rule.

Exercises:
1) A deck of cards is shuffled.
a) Find the chance that the top card is the ace of spades and the bottom card is the ace of spades.

b) Find the chance that the top card is the ace of spades or the bottom card is the ace of spades.

2) A die is rolled 6 times.
a) Find the chance that the first roll is 1 and the last roll is 1.

b) Find the chance that the first roll is 1 or the last roll is 1.

3*) Can we extend the conclusion of addition rule from two things to three things? That is, is the chance that at least one of three things ((i), (ii), and (iii)) will happen = the chance that (i) will happen + the chance that (ii) will happen + the chance that (iii) will happen – the chance that all three things (i), (ii), (iii) will happen? Why?

3. Listing the Ways

When trying to figure chances, it is sometimes very helpful to list all the possible ways that a chance process can turn out. If this is too hard, writing down a few typical ones is a good start.

Example: Two dice are thrown. What is the chance of getting a total of 2 spots?

Solution: To keep the dice separate, imagine that one is white and the other black. How many ways are there for the two dice to fall? The white die can fall in 6 ways: 1, 2, 3, 4, 5, and 6. Similarly, there are 6 possible ways for the black die to fall. So there are $6 \times 6 = 36$ possible ways for the dice to fall. They are all equally likely, so each has 1 chance in 36. There is only one way to get a total of 2 spots: white die 1, black die 1. Therefore, the chance is 1/36.

Exercises:

1) A pair of dice is thrown once.

a) Find the chance of getting a total of 4 spots.

b) Find the chance of getting a total of 10, 11, or 12 spots.

c) Find the chance that both dice show 3 spots.

d) Find the chance that both dice show the same number of spots.

f) Find the chance of getting two different numbers.

2) There are two boxes: box A and box B. In box A, there are three tickets: 1, 2, and 3. In box B, there are four tickets: 1, 2, 3, and 4. One ticket will be drawn at random from each of the two boxes.

a) Find the chance that the number drawn from A is larger than the one from B.

b) Find the chance that the number drawn from A equals the one from B.

c) Find the chance that the number drawn from A is smaller than the one from B.

3*) A pair of dice is thrown 1000 times.
a) What total should appear most often?

b) What totals should appear least often?

4*) Three dice are thrown once. Find the chance that the total number of spots is less than or equal to 9.

5*) (Monty Hall Problem) Suppose you are on a game show, and you are given the choice of three doors: Behind one door is a Master Ball; behind the others, ordinary Poké balls. You pick a door, say No.1, and the host, who knows what is behind the doors, opens another door, say No.3, which has a Poké ball. He then says to you, 'Do you want to pick door No.2?' Is it to your advantage to switch your choice?
(Note: It is a classical problem in Bayesian Statistics, and the formal solution may involve complicated formulas. But actually, you can solve it by listing all the possibilities as well.)

4. Binomial Formula

The binomial coefficient is the number of ways to arrange n objects in a row, when k are alike of one kind and n-k are alike of another. It is usually called 'n choose k' because it gives the number of ways to choose k things out of n, and its value is equal to $n!/(k!(n-k)!)$. (Note: $n!=n\times(n-1)\times\ldots\times2\times1$.)

Example: Find the number of different ways of arranging 2 A's and 3 B's in a row.
Solution: The number of patterns is given by the binomial coefficient: 5!/(2!3!)=10.

Exercises:
1) Find the number of different ways of arranging 1 A's and 3 B's in a row. Write out all the patterns.

2) Find the number of different ways of arranging 2 A's and 2 B's in a row. Write out all the patterns.

Suppose a chance process is carried out as a sequence of independent trials. Then the chance that an event will occur exactly k times out of n is given by the binomial formula: $n!/(k!(n-k)!)\times p^k\times(1-p)^{n-k}$. In this formula, n is the number of trials, k is the number of times the event is to occur, and p is the probability that the event will occur on any particular trial.

Example: A die is rolled 10 times. Find the chance of getting exactly 2 aces.
Solution: $p=1/6$, so the chance = $10!/(2!8!)\times(1/6)^2\times(5/6)^8 \approx 29\%$.

Exercises:
3) A die is rolled 4 times.
a) Find the chance that an ace (one spot) appears exactly once.

b) Find the chance that an ace appears exactly twice.

4) A coin is tossed 10 times.
a) Find the chance that getting exactly 5 heads.

b) Find the chance of obtaining between 4 and 6 heads inclusive.

Challenge Problem

(Lubell–Yamamoto–Meshalkin Inequality) Professor Oak has 802 different Pokémon in his lab. Suppose A_1, A_2, …, A_{31} are subsets of his 802 Pokémon (not necessarily mutually disjoint), and none of the A_i are subsets of each other. For each i, $|A_i|$ is denoted by a_i (the number of Pokémon in subset A_i). Prove that

$$\sum_{i=1}^{31} \frac{1}{802!/(a_i!(802-a_i)!)} \leq 1$$

Reference

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Answers

0: 1) 1/2
2) 2/3

1: 1) (13/52)×(12/51)
2) $(1/6)^3$
3a) $(2/3)^3$ 3b) $(1/3)^3$ 3c) $1-(2/3)^3$
4a) (4/52)×(3/51)×(2/50)×(1/49)
4b) 1-(4/52)×(3/51)×(2/50)×(1/49)
4c) (48/52)×(47/51)×(46/50)×(45/49)
4d) 1-(48/52)×(47/51)×(46/50)×(45/49)
5) 1-(365/365)×(364/365)×(363/365)×… ×(336/365)

2: 1a) 0 1b) 1/52+1/52-0
2a) 1/36 2b) 1/6+1/6-1/36
3) No. For example, if a die is rolled 3 times, the chance of getting at least one ace is not $1/6+1/6+1/6-(1/6)^3$.

3: 1a) 3/36 1b) 6/36 1c) 1/36 1d) 1/6 1f) 1-1/6
2a) 3/12 2b) 3/12 2c) 6/12
3a) 7 3b) 2 and 12
4) $(1+3+6+10+15+21+28)/6^3$
5) After listing all the possibilities, we can find that the chance of winning will increase from 1/3 to 2/3 if we switch our choice.

4: 1) 4. ABBB, BABB, BBAB, BBBA
2) 4!/(2!2!)=6. AABB, ABAB, ABBA, BAAB, BABA, BBAA
3a) $4!/(1!3!)\times(1/6)^1\times(5/6)^3$
3b) $4!/(2!2!)\times(1/6)^2\times(5/6)^2$
4a) $10!/(5!5!)\times(1/2)^5\times(1/2)^5$
4b) $10!/(5!5!)\times(1/2)^5\times(1/2)^5+2\times10!/(4!6!)\times(1/2)^4\times(1/2)^6$

CP: Randomly put Professor Oak's 802 Pokémon in a row. Define E_i as the event that the first a_i Pokémon in the row is the same as Pokémon in set A_i. The chance that event E_i will happen is equal to $1/(802!/(a_i!(802-a_i)!))$. Since none of the A_i are subsets of each other, E_j and E_k cannot both happen for $j \neq k$. Therefore, $1 \geq \text{Prob}(E_1, E_2, \ldots \text{ or } E_{31} \text{ will happen}) = \sum_i \text{Prob}(E_i) = \text{LHS}$.