WUSTL Math Circle

Mysteries of the Floor Function

For every real number x, the *floor* of x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. For example $\lfloor 7.4 \rfloor = 7$.

1. Compute the following floors:

$$\lfloor 2.6 \rfloor =$$

$$\lfloor 3 \rfloor =$$

$$\lfloor 7.5 \rfloor =$$

$$\lfloor \pi \rfloor =$$

$$\left\lfloor \sqrt{7} \right\rfloor =$$

$$\lfloor -5.6 \rfloor =$$

$$\left\lfloor -\sqrt{5} \right\rfloor =$$

2. Compute the quantity

$$f(n) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n+3}{6} \right\rfloor,$$

for the following random integeres $n = 9, 55, 14, -20$.

$$f(9) = \left\lfloor \frac{9}{3} \right\rfloor + \left\lfloor \frac{11}{6} \right\rfloor + \left\lfloor \frac{13}{6} \right\rfloor - \left\lfloor \frac{9}{2} \right\rfloor - \left\lfloor \frac{12}{6} \right\rfloor = 3 + 1 + 2 - 4 - 2 = 6$$

$$f(55) =$$

$$f(14) =$$

$$f(-20) =$$

3. It seems that $f(n)$ is zero for every integer n. Can you prove this?

4. Here is a strategy to prove it. Note that every natural number n is exactly of one of the following forms

$$6k, \quad 6k+1, \quad 6k+2, \quad 6k+3, \quad 6k+4, \quad 6k+5,$$

where k is an integer.

Assuming $n = 6k$, we compute

$$\begin{aligned} f(n) &= f(6k) \\ &= \left\lfloor \frac{6k}{3} \right\rfloor + \left\lfloor \frac{6k+2}{6} \right\rfloor + \left\lfloor \frac{6k+4}{6} \right\rfloor - \left\lfloor \frac{6k}{2} \right\rfloor - \left\lfloor \frac{6k+3}{6} \right\rfloor \\ &= \lfloor 2k \rfloor + \left\lfloor k + \frac{2}{6} \right\rfloor + \left\lfloor k + \frac{4}{6} \right\rfloor - \lfloor 3k \rfloor - \left\lfloor k + \frac{3}{6} \right\rfloor \\ &= 2k + k + k - 3k - k \\ &= 0. \end{aligned}$$

Now assume $n = 6k + 1$, and compute

$$f(n) = f(6k+1) =$$

5. You can treat the other cases $n = 6k+2$, $n = 6k+3$, $n = 6k+4$, $n = 6k+5$ at home to make sure that $f(n)$ is zero for all integers n.

6. Here are some more identities, all discovered by Indian mathematicien Srinivasa Ramanujan. You might like to try to prove them when you get older.

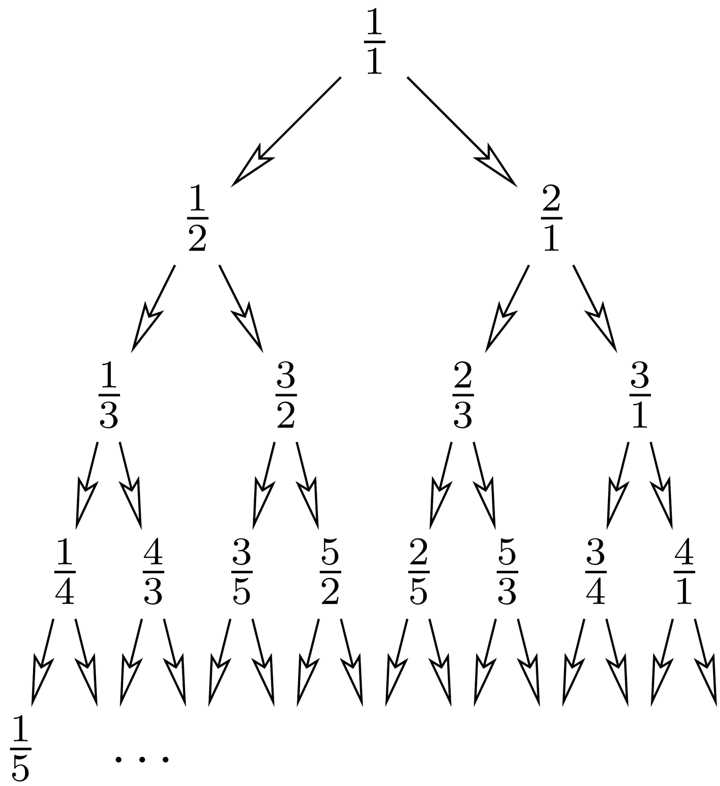
$$\left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{2}} \right\rfloor = \left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{4}} \right\rfloor .$$

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor .$$

Turn the page.

7. Consider the following infinite binary tree of positive rational numbers[1] which

- $\frac{1}{1} = 1$ is on top of the tree, and
- every node $\frac{a}{b}$ has two sons: the left son is $\frac{a}{a+b}$ and the right son is $\frac{a+b}{b}$.



Complete three more rows of this tree.

[1] Picture taken from Martin Aigner and Günter Ziegler, *Proofs from THE BOOK*, Fourth Edition, Springer Verläg, 2010, page 107.

8. Here is the first interesting fact about this tree:

every positive rational number appears exactly once in this tree.

Those of you who are familiar with the *induction principle*, might like to prove this fact at home following the hints given below.

- All fractions in the tree are reduced, that is, if $\frac{a}{b}$ appears in the tree, then a and b has no common prime factor. (Hint: Do downward induction on the place of rows.)
- Every reduced positive fraction $\frac{a}{b}$ appears in the tree. (Hint: Do induction on $a + b$.)
- Every reduced positive fraction $\frac{a}{b}$ appears at most once. (Hint: Do induction on $r + s$.)

Turn the page.

9. Here is the second strange fact about this tree:

List the nodes of this tree as

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \frac{1}{5}, \ \cdots$$

Then this sequence is generated by applying the function $f(x) = \frac{1}{2\lfloor x\rfloor+1-x}$ *repetitively on* $\frac{1}{1}$*; namely*

$$\frac{1}{1}, f\left(\frac{1}{1}\right), f\left(f\left(\frac{1}{1}\right)\right), f\left(f\left(f\left(\frac{1}{1}\right)\right)\right), \cdots$$

To test this assertion, compute

$$f\left(\frac{1}{1}\right) = \frac{1}{2\lfloor 1\rfloor + 1 - 1} = \frac{1}{2}$$

$$f\left(f\left(\frac{1}{1}\right)\right) =$$

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$$f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(\frac{1}{1}\right)\right)\right)\right)\right)\right)\right)\right)\right) =$$

$$f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(\frac{1}{1}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right) =$$

$$f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(\frac{1}{1}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)\right) =$$

10. The famous Fibonacci sequence is

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots$$

where every term (after the first two) is the sum of the two preceding ones.

Here is a strange explicit formula for the n-th term F_n of the Fibonacci sequence in terms of the floor function.

$$F_n = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{1}{2} \right\rfloor .$$

Use your calculator to compute:

$$F_1 = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right) + \frac{1}{2} \right\rfloor = \lfloor 1.22361 \rfloor = 1$$

$$F_2 = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^2 + \frac{1}{2} \right\rfloor =$$

$$F_{12} = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{12} + \frac{1}{2} \right\rfloor =$$

11. There are much more strange facts about the floor function. If interested you could google each of the followings: *Beatty sequence*, *Mills' constant*, *Lambeck-Moser Theorem*.