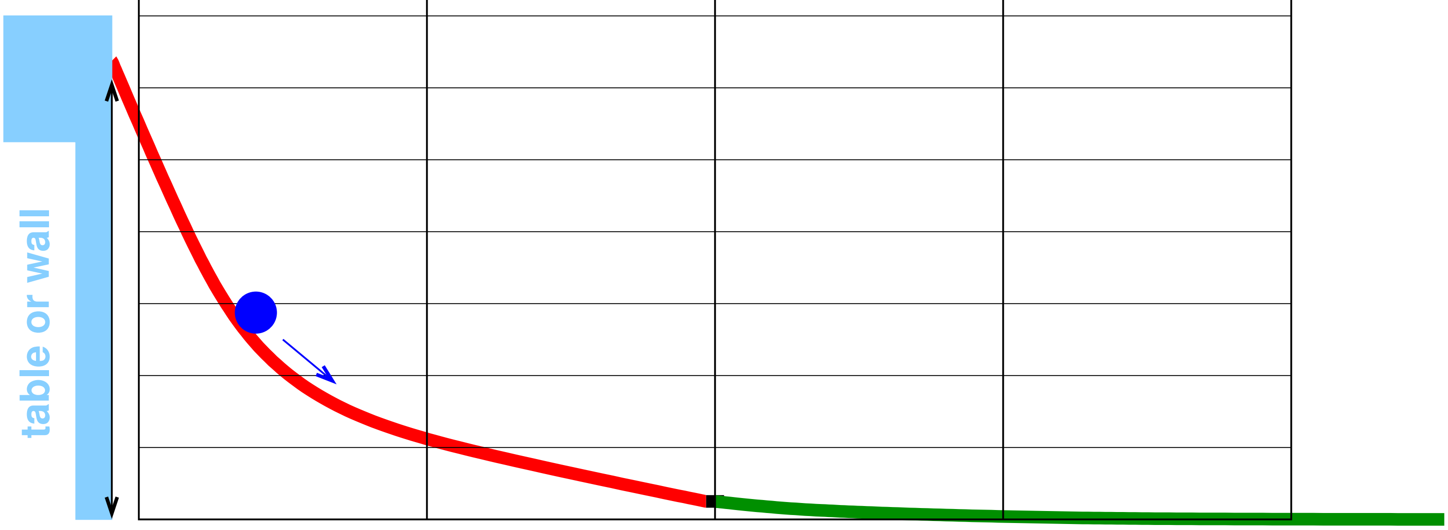
THE MATHEMATICS OF ROLLER COASTERS!!!!!!

I. Rolling downhill

Later in this activity you will make roller coasters (for marbles) out of foam pipe insulation. Let's begin by measuring the speed your marble can attain. You'll need two lengths of pipe, each about 1.8 meters long, some tape, a calculator, a stopwatch, and measuring tape (optional).

Set up the two lengths of pipe as shown, with one acting as a chute and the other mostly flat. (I say "mostly", because you don't want an angle where they are joined.) Record the height of the top of the chute in the first column of the table below. Release the marble, being careful not to push it, from the top of the chute, and time it across the flat length of tube. (Or try it 3 times and average the results.) Record your results in the second column, and divide them to get a speed in the third column. Don't fill out the last column yet.

Starting height (in m)	Time (in s) to cross 2nd tube	Speed (in m/s) across 2nd tube	Computed speed



Before going to the next page, what sort of relationship does your data suggest between height and speed?

II. CONSERVING ENERGY

The abstract concept of energy has roots in the work of Galileo, Leibniz, and Newton, and comes in many flavors. When you release the marble, it begins to lose "potential energy" (inherent in its height) as this is converted into "kinetic energy" (the energy of motion). By the law of *conservation of energy*, the total energy $E = PE + KE$ stays the same.

$$PE_{\text{top}} + KE_{\text{top}} = E = PE_{\text{bottom}} + KE_{\text{bottom}}.$$

Since you are releasing the marble from rest, KE_{top} is zero, and since its height at the bottom is zero, so is PE_{bottom}. What are the other two quantities?

An object feels heavy because of the force of gravity F_{grav}, which is proportional to how "massive" the object is. Gravity tends to make an object accelerate downwards, which means that its speed (in m/s =meters per second) increases. How fast the speed is changing, at least in the absence of wind resistance, is the same for any object: the speed increases by $10m/s$ every second. This requires more force for a more massive object. So $F_{grav} = Mg$, where M is its mass and $g \simeq 10m/s^2$ is the "acceleration due to gravity".

It takes energy to lift an object against this force: if you move it up h meters, then you have imparted Mgh units of potential energy to it. So

$$PE_{\text{top}} = Mgh_{\text{top}},$$

where h is the height in the first column of the table.

Now imagine you are pushing an object into motion, with a constant force, so that it accelerates at a constant rate. When it's accelerating from velocity (speed) $v = 0$ to $5\ m/s$, you don't have to work very hard, because it's not going that far; but when it's going from $v = 5$ to $10\ m/s$, it covers much more ground, and you have to apply the (constant) force over a much greater distance. Even though this takes the same amount of time, it requires much more fuel. The consequence for us is that energy of motion is proportional to the *square* of the speed:

$$KE_{\text{bottom}} = \tfrac{1}{2}Mv^2_{\text{bottom}}.$$

Brain break. Watch the video of how potential and kinetic energy vary in an idealized (frictionless) roller coaster at

https://ninenet.pbslearningmedia.org/resource/hew06.sci.phys.maf.rollercoaster/energy-in-a-roller-coaster-ride

After the energy is "put into" the coaster by the uphill conveyer belt, the total (PE+KE) energy remains constant.

Putting everything together, we have

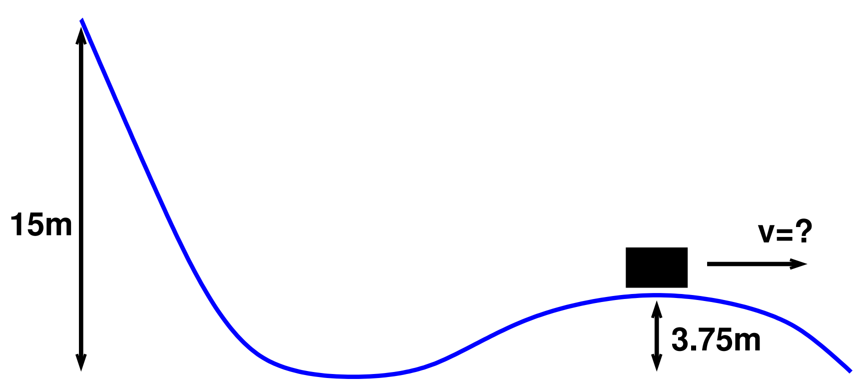
$$Mgh_{\text{top}} = PE_{\text{top}} = KE_{\text{bottom}} = \tfrac{1}{2}Mv_{\text{bottom}}^2,$$

and (doing a little algebra)

$$v_{\text{bottom}} = \sqrt{2gh_{\text{top}}}.$$

Problem 1. Use this (and a calculator) to fill out the last column of the table on the front page. This will probably look rather different from the 3rd column, due to the effects of *friction.* This seems to contradict conservation of energy, but it just means we haven't accounted for the kind of energy (heat!) into which friction converts the kinetic energy. What can you say about the effects of friction based on your data?

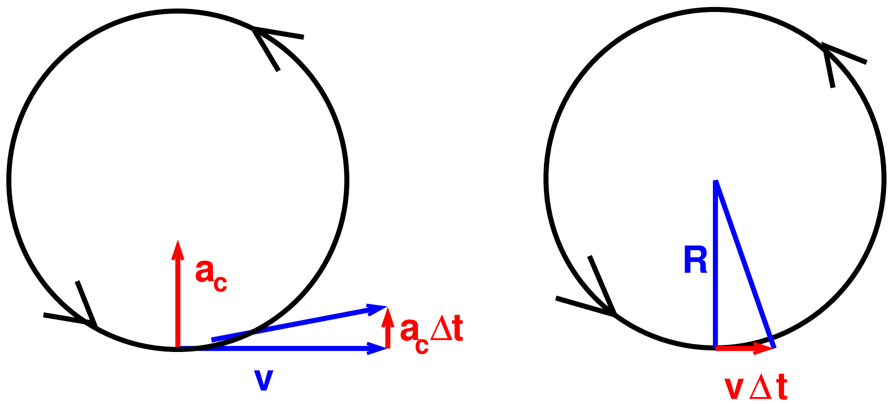
Problem 2. Anna is riding a roller coaster with a drop followed by a hill:



If she started from rest at the beginning of the drop, what is her speed at the top of the hill?

III. Spinning around

When you're moving in a circle, your direction is changing constantly. This requires that force be applied on your body *into the center of the circle*. This force, and the resulting acceleration, are called *centripetal*. Here the acceleration changes your direction rather than your speed. If you think of velocity as an arrow rather than just a number, it all makes sense:



Imagine an object moving in a big circle of radius R at speed v on the ground. In a little interval of time Δt ("change in t"), it moves $v \cdot \Delta t$ meters. The change in the velocity direction is given by the "centripetal acceleration" a_c multiplied by the elapsed time Δt. The triangles are similar, and so the ratios $v\Delta t/R$ and $a_c\Delta t/v$ of side-lengths are the same. We therefore have the key formula

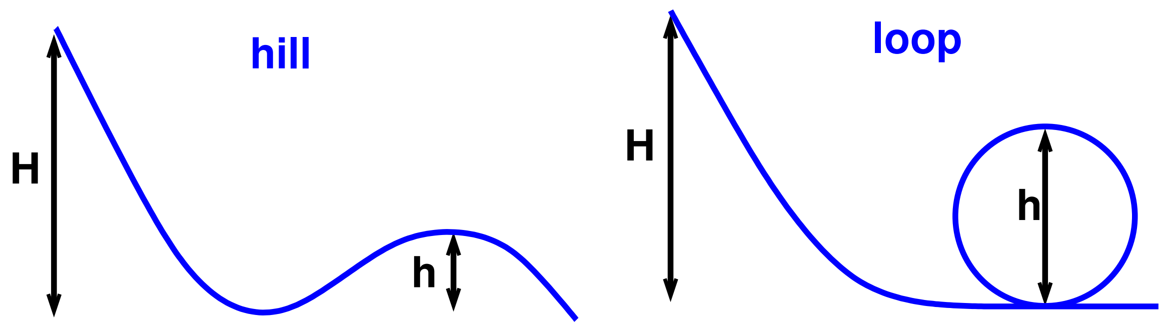
$$a_c = \frac{v^2}{R}.$$

It is helpful to think of acceleration in terms of how many times the acceleration due to gravity it is. So $1g$ is $10m/s^2$, $2g$ is $20m/s^2$, etc. Astronauts are subjected to prolonged continuous $3g$, which is hard on the body. At about 5-6g, especially upward, a normal person will begin to pass out because inertia forces the blood into the legs. This isn't good for business, if your business is designing roller coasters.

Problem 3. There's a ride at Epcot Center called "Mission: Space", which spins a circular arrangement of seats to simulate the experience of takng off in a rocket. Riders face the center of the circle, and feel a "forward" acceleration of $2.5g$. If $R = 9m$, how fast does the operator have to spin the ride to achieve this feeling? (Express your answer in terms of how many seconds it takes for the rider to go through a complete circle; this is called the *period* P.)

IV. "THE SUM OF ALL THRILLS"

This is the name of possibly the most underrated attraction at Epcot, where you get to design your own roller coaster, and then experience it in virtual reality (while attached to the end of a giant robot arm!). What you'll do now is on a much smaller scale, using a single foam pipe (and tape) to form a mini-coaster and trying it out on the marble. Here are two configurations to start with:



By definition, "loop" means a perfect circle.

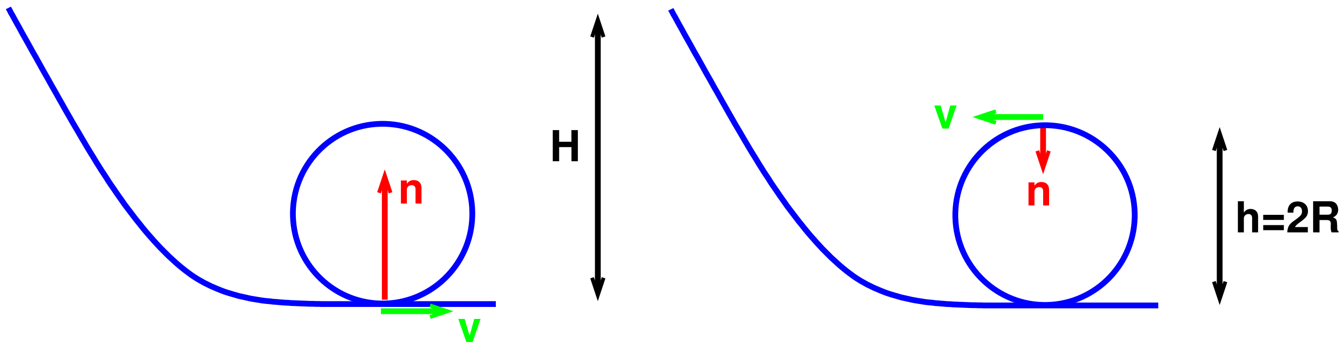
Problem 4. Releasing the marble from rest at the top, how large can you make the ratio h/H and still have the marble reach the end successfully? (Falling out of the loop doesn't count.) Use just one length of tube[1] and start the marble from rest. Does holding the loop rigid (with hands or tape) make a difference?

roller coaster feature	h (height of feature)	H (starting height)	maximal $\frac{h}{H}$
hill			
loop			

[1] there is also a rubber tube; does the material make any difference?

V. STAYING SAFE

In order not to have the roller coaster car (or riders) turn into projectiles (or pass out), we should do the math to understand the forces acting on them.



In both pictures, the "normal" acceleration n is how heavy (in terms of number of g's) the rider feels, i.e. the force exerted by the seat. This isn't a_c: we have to take gravity into account too! In the picture at left, part of n simply counteracts gravity, and we need that much more to get the centripetal acceleration going. That is,

$$n_{\text{bottom}} = \frac{v_{\text{bottom}}^2}{R} + g.$$

On the other hand, at the top, gravity furnishes part of the centripetal acceleration, and the normal force is reduced by that amount:

$$n_{\text{top}} = \frac{v_{\text{top}}^2}{R} - g.$$

(I should also point out that $v_{\text{top}} < v_{\text{bottom}}$.) If n_{top} is negative, that means the rider needs a safety bar to exert that force (in the *outside* direction) to keep him in his seat!

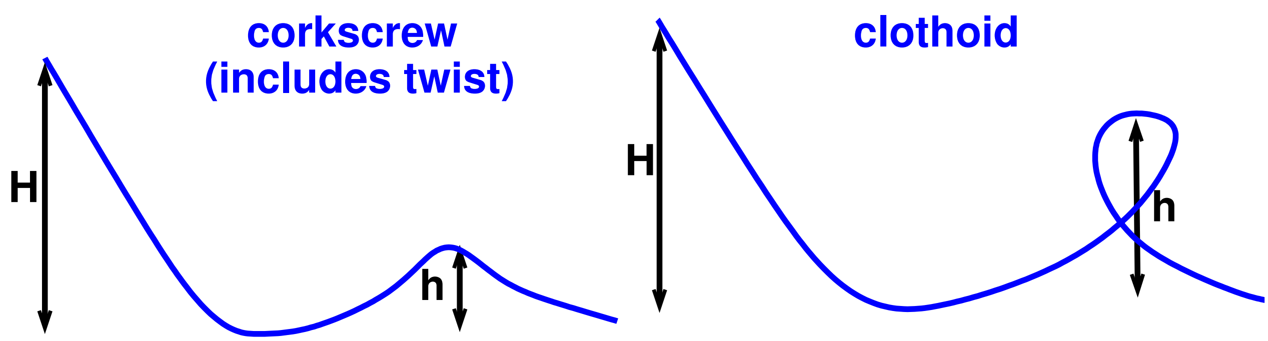
Problem 5. In Problem 2 above, you should have found that Anna was traveling at $15m/s$ at the top of the hill. If the hill is part of a circle with radius $R = 15m$, does she ned a safety bar? [Hint: "reverse" the second calculation.]

Problem 6. Assume there's no friction. If the rider on a loop has a harness and the car has an extra wheel to keep it on the track, then we could take $h = H$. (Why?) Without these things (but still assuming no friction), how high can we build the circular loop (i.e. what is $\frac{h}{H}$)? You'll likely find this is far from the reality of what you were able to achieve with the marble!

VI. Improving your design

Problem 7. Show that in the idealized situation of Problem 6, even if you take h as high as possible ($= \frac{4}{5}H$), so that the rider feels "weightless" at the top, the rider is still subjected to 6 g's at the bottom of the loop. This is unacceptable!! (and also why such loops are not used in real roller coasters)

Returning to the project of designing roller coasters, try some different sorts of roller-coaster features:



(or come up with your own). Fill out rows of the table above for each. Do you get better or worse results than for the strictly circular loop? Can you optimize the design for the marble coaster?

Problem 8. Do a general calculation: if the "bottom" of a feature (at height 0) is part of a circle of radius R, and the top (at height h) is part of a circle of radius r, what is the difference between the normal accelerations at the bottom and top, $d = n_{\text{bottom}} - n_{\text{top}}$? (You should get $2g$ times a function of H, h, R, and r.) Now try plugging in various options:

- the circular loop corresponds to $R = r = \frac{1}{2}h$;
- the corkscrew with length equal to π times h has $R = r = h$;
- one version of the clothoid has $R = h$, $r = \frac{h}{4}$.

We want to minimize the difference d. Which one is best?

Problem 9. Noah is riding a clothoid loop with radius $R = 16m$ at the bottom and $r = 3m$ at the top. Suppose he is going $6m/s$ at the top and $18m/s$ at the bottom. What g-forces will he feel at the top and bottom?