Hotel Hilbert
Washington University Math Circle

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1 Welcome to your new job!

Welcome to your new job at Hotel Hilbert! You and your friends have been hired as the new managers of the hotel, which was opened by famous mathematician David Hilbert in 1924.
Things you should know about the hotel:

1. The hotel has an *infinite* number of rooms! The rooms are numbered increasingly $1, 2, 3, \ldots, 100, \ldots, 9999, 10000, 10001, \ldots$.
2. *Exactly one* guest is allowed in each room.
3. When you room a guest, you must be able to tell where he or she came from and exactly which room he or she is in.
4. You can never kick out a guest.

Things are crazy around here, so get ready to deal with some weird problems!

1.1 Problems

First you must form a management team. These teams must be 2 or 3 people. Come up with a team name– extra 'points' for mathy names.

1. On your first day the hotel is completely full. A dog walks into the hotel and needs a room (of course dogs are welcome!). With your team, come up with a way to find the dog a room. Make sure all of your team members understand the strategy and then sketch the strategy below. Dog drawings encouraged.

2. The dog doesn't like the room you gave her. She wants to be in room 7– her favorite number and a prime! Figure out how to get the dog in room 7. Sketch below.

3. The next day, the dog invites 6 of her friends and requests that they be put in rooms 1 through 6 (she remains in room 7). Accommodate the other dogs and sketch below.

4. The next day the dog and her friends leave. The dog enjoyed the hotel so much that she returns the day after with *all* of her dog friends. She's a really popular dog so she has an infinite number of them (say, friend $1, 2, 3, \ldots$). She would like a room for each of them but the hotel is full with people. Can you find a way to room all the dogs? Sketch below.

5. A few days later, the hotel is empty and an infinite number of buses arrive (numbered $1, 2, 3, \ldots$). On each bus there is an infinite number of people (numbered $1, 2, 3, \ldots$). Rumor has it that this many guests has arrived before, and Mr. Hilbert figured out how to accommodate all of them. Figure out how to do this and sketch below.

6. A few days later, the hotel is *full* and an infinite number of buses arrive with an infinite number of people on each bus. Figure out how to room them all and sketch below.

7. How many rooms are empty after rooming all of the buses?

8. After a few months, and dealing with all the buses, the managers decide to build more bus lanes– an infinite number of them (again, numbered $1, 2, 3, \ldots$)! Once the construction is complete, an infinite number of buses with an infinite number of people show up in *each* lane. Figure out how to room all of the people when the hotel is *empty* and when the hotel is *full*. Sketch below.

2 Infinity

We define the *cardinality* of a set as the number of elements in the set. For example, the set containing the numbers 1 through 10 has cardinality 10. The set containing the letters $a, b, c, \ldots, z$ has cardinality 26.
You may ask, "What is the cardinality of the set of rooms in Hotel Hilbert?"
In general, if a set can be counted as $1, 2, 3, \ldots$, we say the set is countably infinite (like the room numbers of Hotel Hilbert) and say that it has cardinality $\aleph_0$ (said 'aleph naught' where $\aleph$ is the first letter of the Hebrew alphabet).
More rigorously, we say that a set is countably infinite if it can be put in a one-to-one correspondence with the natural numbers ($\mathbb{N}$). That is, if the set is denoted S, there exists a function f from $\mathbb{N}$ to S such that for all m and n in $\mathbb{N}$, if $f(m) = f(n)$ then $m = n$. We say that f is one-to-one.

2.1 (Challenge) Problems

1. Come up with $2n$ examples of sets that have cardinality $\aleph_0$, where n is the number of managers in your team.

2. Show that the positive rational numbers are countably infinite.

3. Let $|\cdot|$ denote the cardinality of a set, i.e. $|\{a, b, c\}| = 3$, $|\mathbb{N}| = \aleph_0$, etc. Let f be a function from a set A to a set B. Prove that if $|A| > |B|$ then f cannot be one-to-one.