The Birthday Problem

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1 Introduction

Suppose you go to a birthday party for one of your friends, Summer. It's a big party, and there are 25 kids at the event including the birthday girl. Now imagine you and the other partygoers decide to play a game where you all list your birthdays. Before you start, each of you makes a guess whether two people at the party will share the same birthday. If you are correct, you win. So the question becomes how likely it is that two people out of the group of 25 will share the same birthday. What about the different, but related question of whether anybody has the same birthday as Summer, May 11th?

Activity: Let's try this right now! We'll go around the room and record each person's birthday on the board. At the end, we'll be able to see if anybody shares a birthday.

Both of these problems are very mathematical in nature. In fact, both are examples of problems in the mathematical fields of *probability* and *combinatorics*. You are probably familiar with the use of the term "probability" to describe how likely (or unlikely) something is to occur. In mathematics, we make the idea of "likely" precise. On the other hand, *combinatorics* is a term with which you might have less familiarity. It can be described in the simple terms as the mathematical study of counting.

Now suppose we are conducting an experiment (any process where an observation is made, such as the birthday, is referred to as an "experiment" in probability lingo). Possible outcomes that can occur in the experiment and cannot be broken up into simpler outcomes are called *simple events*. The set of all simple events is called the *sample space*. More generally, an *event* is a collection of simple events in the sample space.

The basic idea of computing the probability of Event A of using counting methods from combinatorics is as follows:

1. Count the total number of simple events that can occur in the experiment. Call this number n.

2. Count the number of simple events in Event A. Call this number m.

3. The fraction $\frac{m}{n}$ gives the probability of event A occuring.

Soon we will turn to using this method to solve the birthday problem. First, though, let's work some simpler counting and probability problems.

2 Warm-Up

Here are a few simpler questions to warm up.

Suppose Fred has five pairs of shirts, four pairs of pants, and three pairs of shoes. All the clothing items are different. Exactly one of Fred's shirts is yellow and exactly one of Fred's pants is red. Fred will wear any combination of the clothing items, except he will not wear the yellow shirt with the red pants. How many outfits (a combination of shirts, pants, and shoes) can Fred wear? Pictures encouraged!

Another one of the five shirts is red. Assuming Fred is equally likely to choose any of the outfits, what is the probability that Fred chooses an outfit with a red shirt?

A pair of dice is rolled and their sum is recorded. What is the probability of rolling a 7?

3 Permutations, Combinations, and Probability

We now turn to the study of *permutations* and *combinations*. Let's start with some simple questions. Recall $n! = n \times (n-1) \times \cdots \times 2 \times 1$- try to write your answers using factorials!

In a class of 20 students, how many ways are there for a teacher to choose 5 students sequentially to come to the board to work problems (counting order)?

Suppose we know the teacher calls Christine, Aaron, Cole, Taylor, and Nathan to the board. How many different orders are there in which can the students be called?

In a class of 20 students, how many ways are there for a teacher to form a group of 5 students (here order doesn't matter)?

In general, how many ways are there to pick an ordered arrangement of k items from a set of n objects? We refer to these orderings as k-permutaions of n and write $P(n,k)$ to denote the number of such arrangements.

How many ways are there to order k items?

How many ways are there to select k items from a set S of n objects (not counting order)? We say this is the number of k-combinations of S. This number is denoted $\binom{n}{k}$, read "n choose k." These numbers are also called *binomial coefficients*.

Consider the algebraic expression $(1 + x)^n$, with $n \geq 2$. Imagine we expand this expression out. What is the coefficient of x^{n-1}? Hint: if you are having trouble, try it for $n = 2$ and see if you can figure out what's going on.

What is the coefficient of x^{n-2}?

In general, what is the coefficient of x^j, $0 \leq j \leq n$?

Give an argument that $\binom{n}{k} = \binom{n}{n-k}$

Challenge Problems

Show that $\sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$. Extra challenge: show this two different ways.

There are thirty people at a party. They form three distinct groups of 10, A, B, and C. People in group A know everybody else in group A, but nobody in group B or C. People in group B only know people in group C, and people in group C only know people in group B. People that know each other hug, and people that don't know each other shake hands. If two people from the party are randomly chosen, what is the probability that they will greet each other with a hug?

Five cards are dealt from a standard 52-card deck. What is the probability we draw

(a) 3 aces and 2 kings?

(b) a "full house" (3 cards of one kind, 2 cards of another kind)?

A balanced die is tossed six times, and the number on the uppermost face is recorded each time. What is the probability that the numbers recorded are $1, 2, 3, 4, 5, 6$ in any order?

Show that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. Extra challenge: show this two different ways.

4 The Birthday Problem

At last we come to the birthday problem! With the tools we've developed, we can actually compute specific probabilities for the events we discussed earlier.

Consider a room full of 20 people. Suppose we number each person $1 - 20$. Next to each number, we write the person's birthday. How many outcomes are there in this experiment?

Of those outcomes, how many of them have nobody sharing a birthday?

What is the probability that no two people in the group of 20 share a birthday? What about the probability that at least two people share a birthday?

Repeat these questions, but with a group of n people instead of a group of 20 people. Write down a formula $P(n)$ for the probability that in a group of n people, at least two people share the same birthday (here $n < 365$).

Suppose you are part of a group of n people. What is the probability that at least one other person in the group shares *your* birthday?

Fun Fact: If there are just 23 people in the group, the probability that at least 2 people shares the same birthday is approximately 50 percent!!

Challenge Problems

Give a expression for $P(n)$ using binomial coefficients.

Use the approximation $e^x \approx 1 + x$ for small x to write an approximate formula of $P(n)$ using the exponential function. (Hint: take x negative). Then graph this function.

5 An Extra Challenge

Here is an additional fun, but tricky problem in combinatorics.

How many ways are there to write 100 as a sum of 7 positive integers? Let's assume order matters; for example, $50 + 25 + 10 + 5 + 4 + 3 + 3$ is counted distinctly from $3 + 3 + 4 + 5 + 10 + 25 + 50$.

In general, how many ways are there to write an integer n as the sum of k positive integers (here $k \leq n$)?