The Game of Set

Feb 11th, 2018

1. Let $S$ denote a standard SET deck. $S$ contains one of each possible type of card. How many cards are there in $S$?

2. How many sets are there in $S$?

3. Prove the Fundamental Theorem of SET:

   **Theorem 1** Given two cards, $x, y \in S$; there is a unique card $z \in S$ for which $(x, y, z)$ is a set.

4. Given two set cards, $x$ and $y$, define their **product**, $x \star y$, to be the unique card for which $(x, y, x \star y)$ is a set.
   
   What properties does this multiplication have? Is it commutative? Is it associative? Is there an identity?
5. There should be a correspondence between $S$ and the integers $\{1, 2, ..., 81\}$ by writing the numbers in base 3. How does this correspondence work?

   Hint: Another way to think of this is each card can be though of as a vector $(x_1, x_2, x_3, x_4)$ where each of the $x_i$ are either 0,1 or 2. (How is this the same as thinking in base 3?)

6. By writing elements in $S$ as vectors (or as numbers between 0 and 80 in base 3), determine the set product for these numbers.

   Hint: look at many examples and look at what happens in each component of the vector.

   (The set of vectors above is called $\mathbb{Z}_3^4$).

7. Compute the following products:
   1. $(1, 0, 0, 0) \star (1, 0, 0, 1) = $
   2. $(0, 1, 0, 0) \star (0, 2, 0, 1) = $
   3. $(2, 2, 2, 2) \star (0, 2, 2, 1) = $
   4. $(0, 0, 0, 0) \star (1, 1, 1, 1) = $

8. Given a card $x \in S$, how many sets does $x$ belong to?

9. Does your answer in Question 8 agree with your answer in Question 2?
10. What is the smallest stack such that every card must have a set in the stack? In other words, you want to deal a number of cards, \( n \), so that you know, without looking at the cards, that every one of those cards is in at least one set. How big must \( n \) be?

11. Now, what is the largest stack with no sets?

Hint. First consider the game of Set with only 2 properties (9 cards in the deck), say shape and number.

12. (Open Question) You can generalize the game of SET by adding more Features, say \( k \); or adding more options in each feature, say we used \( n \) shapes, colors, shadings, etc. Answer the rest of the questions with this new deck of Set cards. (Many of the questions, such as Question 11 have not been answered in this general situation.)