Fantastic Factoring

Following are some factoring patterns that you might already recognize. $x$ and $y$ can both represent variables in the expressions, or $y$ might be a constant. These rules work for all real numbers $x$ and $y$. Sometimes you are given the factored form and recognizing the pattern will save you time and errors in not multiplying out all of the terms.

**Difference of Squares**

$x^2 - y^2 = (x - y) (x + y)$

**Binomial Squares**

$x^2 - 2xy + y^2 = (x - y) (x - y)$

$x^2 + 2xy + y^2 = (x + y) (x + y)$

**Difference of Cubes**

$x^3 - y^3 = (x - y) (x^2 + xy + y^2)$

**Sum of Cubes**

$x^3 + y^3 = (x + y) (x^2 - xy + y^2)$

Factor the following expressions.

1. $64a^4 - 100b^4$
2. $27m^3 + 8$
3. $4b^2 - 36bc + 81c^2$

By looking at the sum and difference of cubes above, how do you think the general difference and sum below would be factored?

$x^n - y^n =

For all odd $n$, $x^n + y^n =

Why does the sum only work for odd $n$?

Often an expression is not a binomial or trinomial and has 4 or more terms. We use different methods to factor these. For example, let’s use **factoring by grouping** to simplify and solve.

4. $4ab - 8b^2 + 3a^3 - 6a^2b$
5. $xy + y + x + 1 = 0$
The entire purpose of factoring a polynomial is to help in simplifying and solving polynomial equations. Most problems you have probably seen have set the equation equal to 0 to solve; however, if looking for integer solutions, this doesn’t always have to be the case.

6. If \( x \) is a positive integer and \( x(x + 1)(x + 2)(x + 3) + 1 = 379^2 \), compute \( x \).

What if factoring by grouping doesn’t work? **Simon’s Favorite Factoring Trick** to the Rescue! SFFT allows you to think about the problem algebraically or visually by completing the rectangle.

Example: Given that \( j \) and \( k \) are integers and \( j^2 + 5j^2k^2 - 20k^2 = 109 \), find \( 5j^2k^2 \).

7. Both \( p \) and \( q \) are positive integers where \( p > q \). Find all ordered pairs \((p, q)\) such that \( 2pq + 2p - 3q = 18 \).

8. Twice the area of a non-square rectangle equals triple its perimeter. If the dimensions are both positive integers, what is the rectangle’s area?
9. Compute all integer values of \( n \), \( 90 \leq n \leq 100 \), that can not be written in the form \( n = a + b + ab \), where \( a \) and \( b \) are positive integers.

10. Compute the positive integer \( x \) such that \( 4x^3 - 41x^2 + 10x = 1989 \).

11. If \( x^5 + 5x^4 + 10x^3 + 10x^2 - 5x + 1 = 10 \) and \( x \neq -1 \), compute the numerical value of \( (x + 1)^4 \).

12. Let \( A, M, \) and \( C \) be nonnegative integers such that \( A + M + C = 12 \). What is the maximum value of \( A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A \)? (Hint: Look back at question #5 and its solution.)
13. Find the number of ordered pairs of integers \((m, n)\) for which \(mn \geq 0\) and \(m^3 + n^3 + 99mn = 33^3\) is true.

14. Let us examine the expression \(a^3 + b^3\), where \(a > b\). One well-known result is that \(a^3 + b^3 = c^3\) has no solution in positive integers. For each of the equations below, either:
   1. Prove that no solutions can exist  OR
   2. Show how an infinite number of solutions can be generated.

   A. \(a^3 + b^3 = c^2\)

   B. \(a^3 + b^3 = c \cdot d \cdot e\), where \(c\), \(d\), and \(e\) are in geometric progression

   C. \(a^3 + b^3 = c \cdot d \cdot e\), where \(c\), \(d\), and \(e\) are in arithmetic progression

   D. \(a^3 + b^3 = 3p\), where \(p\) is a prime greater than 3
Solutions to Fantastic Factoring

1. \(4(4a^2 - 5b^2)(4a^2 + 5b^2)\)
2. \((3m + 2)(9m^2 - 6m + 4)\)
3. \((2b - 9c)^2\)
4. \((a - 2b)(4b + 3a^2)\)
5. \(x = -1\) or \(y = -1\)
6. 18 (1989 ARML, Individual #1)
7. (4, 2)
8. 48
9. 96 and 100 (1990 ARML, Team #7)
10. 13 (1989 NYSML, Individual #2)
11. 10 (1994 ARML, Team #1)
12. 112 (2000 AMC, #12)
13. 35 (1999 AHSME, #30)
14. A. Infinite number of solutions
   B. No solutions
   C. Infinite number of solutions
   D. No solutions
   (1990 ARML PQ Part I)

Team Round Answers

1. 6481 (1992 NYSML, Team #5)
2. 186 (mathleague.org 11207, Large Team #4)
3. 2013 (mathleague.org 11607, Team #2)
4. 6 (mathleague.org 11301, Sprint #10)
5. ± 3i (1991 NYSML, Individual #2)
6. 1600 (mathleague.org 11202, Large Team #7)
7. -61 (AHSME 1966, #30)
8. -403 (mathleague.org 11607, Target #6)
9. 4 (mathleague.org 11308, Sprint #28)
10. 96 (mathleague.org 11307, Sprint #11)
Team Round
30 minutes - 10 questions - maximum of 6 team members
There is no penalty for guessing.

1. The number \((9^6 + 1)\) is the product of three primes. Compute the largest of these primes.

2. Of the integers between 1 and 2310, how many are divisible by exactly three of the five primes 2, 3, 5, 7, and 11?

3. If \(x\) and \(y\) are positive integers such that \(x^2 = y^2 + 61\), find \(x(x + 2) + y(y + 3)\).

4. The graph of \(xy + 3x + 2y = 0\) can be produced by translating the graph of \(y = \frac{k}{x}\) to the left and down for some constant value \(k\). Find \(k\).

5. Let \(f(x) = x^2 + bx + 9\) and \(g(x) = x^2 + dx + e\). If \(f(x) = 0\) has roots \(r\) and \(s\), and \(g(x) = 0\) has roots \(-r\) and \(-s\), compute the two roots of \(f(x) + g(x) = 0\).

6. How many ordered pairs of integers \((x, y)\) with \(1 \leq x \leq 100\) and \(1 \leq y \leq 100\) make the quantity \(xy - x - y\) a multiple of 5?

7. If three of the roots of \(x^4 + ax^2 + bx + c = 0\) are 1, 2, and 3, find the value of \(a + c\).

8. \(x\) and \(y\) are real numbers that satisfy the equations \(x - y = 1\) and \(x^5 - y^5 = 2016\). Find \(\frac{x^5 + y^5}{x + y} - (x^4 + y^4)\).

9. How many ordered pairs of positive integers \((a, b)\) are there such that \(\frac{1}{a} - \frac{1}{b} = \frac{1}{143}\)?

10. Suppose that \(a, b, c, d\) are real numbers such that \(ab + 3a + 3b = 216, bc + 3b + 3c = 96, cd + 3c + 3d = 40\). Find the maximum possible value of \(ad + 3a + 3d\).