Fun with Combinatorial Games

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Note: This material was adapted from Thomas S Ferguson's Game Theory

Introduction

Consider the following simple game with two players: There is a pile of 21 chips. Each players takes turns removing chips from the pile, but they can only remove 1,2, or 3 chips at a time. The game ends when there are no more chips in the pile, and the last player to remove chips wins. We'll call this game the Take-Away Game. Try playing this game with your neighbor (no need to use actual chips- you can just represent the chips as a number on your paper). Is there a strategy to this game? Does the first or second player have an inherent advantage? Does neither player have an advantage?

You probably have played various types of games before, but have you ever considered how mathematical games are? From fairly complicated games such as chess to very simple games such as tic-tac-toe, mathematics plays a role. In particular, math can tell us information about whether one player or the other has a *winning strategy*. We know that tic-tac-toe, for example, will end in a draw if both players play optimally (in others words, both players do not make any strategic mistakes). But what about a game such as checkers? It is certainly a step up in complexity from tic-tac-toe, but it still seems relatively simple. It turns out that it wasn't until 2007 that it was shown, with the help of computers, that the game of checkers ends in a draw if both players play optimally. We say the game was *solved*. Chess, which is of course substantially more complicated than checkers, still remains unsolved, and probably will remain so for the foreseeable future!

In this math circle, we focus on games that CAN be solved, without the aid of supercomputers. In particular, we focus on simple games where there is always a winner and both players have perfect information. Such games are called *combinatorial games*. Formally, we define a combinatorial game according to the following characteristics:

Definition 1 (Combinatorial Game)**.**

1. There are two players.
2. There is a set, usually finite, of possible positions of the game.
3. The rules of the game specify for both players and each position which moves to other positions are legal moves. If the game makes no distinction between the two players, it is called impartial; otherwise, it is called partizan.
4. The players alternate moving.
5. The game ends when a position is reached from which no moves are possible for the player whose turn it is to move. Under *normal play*, the last player to move wins, while under *misère play* the last player to move loses.
6. The game ends in a finite number of moves no matter how it is played.

We see that the Take-Away Game described previously is an example of a combinatorial game. Now let's analyze it a bit more.

The Take-Away Game, N and P positions

Exercise 1: Let's return to the Take-Away Game. Suppose there are 4 chips left. Which player has an advantage: the previous player (who just moved), or the next player? Explain your answer.

Exercise 2: Suppose there are 7 chips left. Which player has the advantage?

Exercise 3: Suppose there are 8 chips left. Which player has the advantage?

Exercise 4: Conjecture (this means make a guess) on a pattern for which positions are P *positions* (positions which are winning for the previous player) and which positions are N *positions* (positions which are winning for the next player). Then prove your conjecture. This should tell you whether the first or second player in the Take-Away Game has a winning strategy.

In fact, we can use the following general algorithm for determining which positions in a combinatorial game are N positions and P positions, and hence solve the game:

1. Label every terminal position as a P position.
2. Label every position that can reach a labeled P-position in one move as an N position.
3. Find those positions whose only moves are to labeled N-positions; label such positions as P-positions.
4. If no new P-positions were found, stop. Otherwise, return to step 2.

In fact, N positions and P positions are characterized by the following properties:
1. All terminal positions are P positions.
2. From each N position, there is a move to a P position.
3. Every move from a P position is to an N position.

Exercise 5: Let's quickly check your understanding. Explain why in a combinatorial game, either the first player or the second player has a winning strategy.

Let's return to the Take-Away game.

Exercise 6: What are the N positions and P positions of the Take-Away game if the rules are changed from ordinary play to misère play?

Exercise 7: Suppose there are 50 chips rather than 21 and players can remove 1, 2, 3, or 4 chips. Determine the N positions and P positions. Which player has the advantage?

The Game of Chomp

We now consider another combinatorial game, this one called *Chomp*. Chomp is played on an $m \times n$ grid. Imagine a chocolate bar with one poison piece in the lower left hand corner. Each player takes turns "chomping" on the bar; he or she selects a square in the grid, and then all the squares to the right and above that square (including the square selected) are removed. The player that is forced to eat the poisoned square loses.

Exercise 8: Try playing Chomp with a neighbor on the 3×7 board provided (see Figure 1 on the handout). Jot down any observations you have.

Exercise 8: Determine whether each of the following configurations of Chomp is an N position or P position. (see Figure 2 and Figure 3 on handout. If it is an N position, find the winning move.

Exercise 9: Prove that the first player in Chomp always has a winning strategy (Hint: argue by contradiction. Suppose the first player does not have a winning strategy; then the second player must have a winning strategy. Suppose the first player just removes the top right square of the bar and try to reach a contradiction).

The Game of Nim

We now turn to a famous combinatorial game that is more complicated than the Take-Away Game but similar in spirit- the Game of Nim. This game has been around for a long time; perhaps dating back to ancient China! The game consists of three piles of chips, say of size x_1,x_2, and x_3. Players alternate removing chips from the piles. Each player can remove as many chips as they wish, but only from a single pile at a time. Under normal play, the last player to remove chips wins. Try playing Nim with your neighbor with piles of size 5,7, and 9.

Exercise 10: Now let's try to analyze the game. Suppose there are chips left in only one pile. What are the N positions and P positions?

Exercise 11: Now suppose there are two piles with chips remaining. What are the N positions and P positions in this situation?

Exercise 12: Notice if there are three non-empty piles, the situation becomes more complicated. Try to determine N and P positions for small numbers of chips in three piles: say positions $(1,1,1)$, $(1,1,2)$, $(1,1,3)$,$(1,2,3)$ and $(1,2,2)$. Here an ordered triple (x_1,x_2,x_3) is shorthand for the position with x_1 chips in the first pile, x_2 chips in the second pile, and x_3 chips in the third pile.

The full solution to the game of Nim was discovered by Louis Bouton in 1902 and this result was considered to be the birth of combinatorial game theory. Fun fact: Bouton actually got a Master's degree at Washington University! Before we examine Bouton's solution, which is simple and elegant yet quite clever, we need to review base-2 decimal expansions and define something called a *Nim Sum*. Recall that any positive integer k can be written in base-2 as $k = a_n 2^n + a_{n-1} 2^{n-1} + \cdots + a_1 2 + a_0$, where $a_j = 0$ or 1 for each j. We write the number in its base-2 representation as $(a_n a_{n-1} \ldots a_1 a_0)_2$.

Exercise 13: Express the number 37 in base-2.

Also recall arithmetic modulo 2- here $0 + 0 = 0$, $1 + 0 = 1$, and $1 + 1 = 0$. Arithmetic modulo 2 is really just writing a 0 for an even number and 1 for an odd number and using the rules that the sum of two odd or two even numbers is even, while the sum of one odd and one even number is odd.

Next we define something called a *nim-sum*. In informal language, we take two numbers expressed base-2 and add each digit individually mod 2 to obtain a third number expressed in base-2.

Definition: The *nim-sum* of $(x_m \ldots x_0)_2$ and $(y_m \ldots y_0)_2$ is $(z_m \ldots z_0)_2$, where for all k $z_k = x_k + y_k \pmod{2}$. We write $\mathbf{x} \oplus \mathbf{y} = (x_m \ldots x_0)_2 \oplus (y_m \ldots y_0)_2$ to denote the nim-sum.

Exercise 13: Calculate the nim-sum of 13, 12 and 8.

Exercise 14: Explain why if $\mathbf{x} \oplus \mathbf{y_1} = \mathbf{x} \oplus \mathbf{y_2}$, we must have $\mathbf{y_1} = \mathbf{y_2}$.

Bouton proved the following nice result relating the nim-sum to the P positions in Nim.

Theorem 2 (L. Bouton)**.** *A position* $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ *is a* P*-position if and only if the nim-sum of its components is zero, i.e* $\mathbf{x_1} \oplus \mathbf{x_2} \oplus \mathbf{x_3} = 0$.

Exercise 15: Using Bouton's theorem, determine whether the first player or second player has a winning strategy if Nim is played with piles of size 5,7 and 9.

Exercise 16: Prove Bouton's theorem. You should do this by showing that the characterization of N and P positions on page 3 is satisfied. I suggest the following steps:
1. Show all terminal positions are P positions (this is straightforward).
2. Show that from each N position, there is a move to a P position (hint: form the nim-sum as a column addition, and consider the leftmost column with an odd number of 1s).
3. Show that every move from a P position is to an N position (hint: argue by contradiction and use the cancellation law).

Exercise 17: Consider the position above with piles of size 13,12, and 8. What are three winning moves for the next player?

Exercise 18 (challenge): Suppose Nim is played using Misere play rather than ordinary play. Determine a winning strategy.

Exercise 19 (Nimble): Nimble is played on a line of n squares labeled from 0 to n. A finite number of coins are placed on the squares, with possibly multiple coins on one square. Players alternate moving and can move precisely one coin any number of squares to the left (moves onto or over squares with other coins are permitted). The game ends when all coins are on square 0 and the last player to move wins. Explain why this game is just ordinary Nim in disguise and determine if the following position is an N position or a P position (see Figure 4). If it is an N position, determine the winning move.

Exercise 20 (Staircase Nim): Consider the following game. Coins are placed on n-steps (say x_j coins on step j). Players alternate moving and are allowed to move any number of chips on a single step to the step immediately below. So if they move k chips from the jth step, which initially had x_j chips, there will be $x_j - k$ chips on the jth step and $x_{j-1} + k$ chips on the $j - 1$ step after the move. Coins reaching step 0 are removed from play. The last player to move wins. Show that $(x_1, \ldots, x_n)$ is a P position if and only if the numbers of coins on the odd-numbered steps $(x_1, x_3, \ldots, x_k)$, where $k = n$ if n is odd and $k = n - 1$ if n is even, is a P position in ordinary Nim.