MO-ARML --- January, 2019 -- GEOMETRY OF COMPLEX NUMBERS

Many Complex Number problems are most easily solved geometrically on the complex Argand Plane.

<u>GOAL:</u> Treat these as 10-second Mental Arithmetic (or geometry) problems: Simplify $\left(\frac{1+i\sqrt{3}}{2}\right)^3$ *and* $(1+i)^6$

First, the **BASICS**

1. Solve: $x^2 = 25$, x = ________; $x^2 = 7$, x = ______; $x^2 = -9$, x = ______; $x^2 + x + 1 = 0$; x = ____________

DEFINITIONS Define: $i = \sqrt{-1}$ *and* $i^2 = -1$. If a and b are Real numbers, then **z = a + b*i*** is a *Complex Number in rectangular form.* The <u>modulus</u> or <u>*absolute value of z*</u> or <u>*length of vector z*</u> is: $|z| = r = \sqrt{a^2+b^2}$

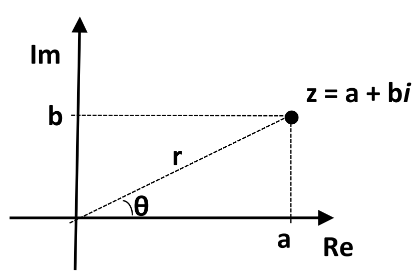
2. Simplify: **(7 – 6*i*) + (5 + *i*);** **(– 8 + *i*) – (– 3 + 4*i*);** **(7 – 6*i*)(5 + *i*);** **(5 + 5*i*)² ;** $\frac{5-2i}{4+5i}$

3. If **z = a + b*i*,** then the *conjugate of z* is $\bar{z}$ = **a – b*i*** .

<u>Theorems</u>: $z \cdot \bar{z}$ = __________ = ________ = ________ **and** $\frac{z}{\bar{z}} =$ ________

<u>Argand Diagram</u> Any *Real Number* can be represented as a <u>point on a Real Number **Line**</u> [one-dimensional].

Any *Complex Number* can be represented as a <u>point on the Complex **Plane**</u> [two-dimensional].



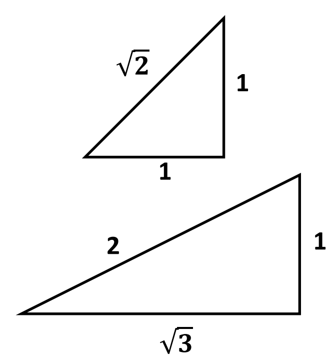
z = a + b*i* *rectangular form*

z = r cos θ + *i* r sin θ *trig or polar form*

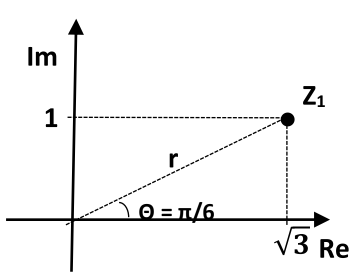
z = r (cos θ + *i* sin θ) = r *cis* θ *'engineer' form*

$z = r\,e^{i\theta}$ *Euler's form*

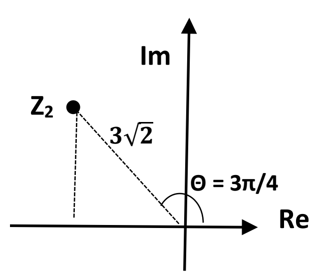
4. Write each complex number in <u>both</u> rectangular and Euler form. **<u>Hints</u>**:



A.



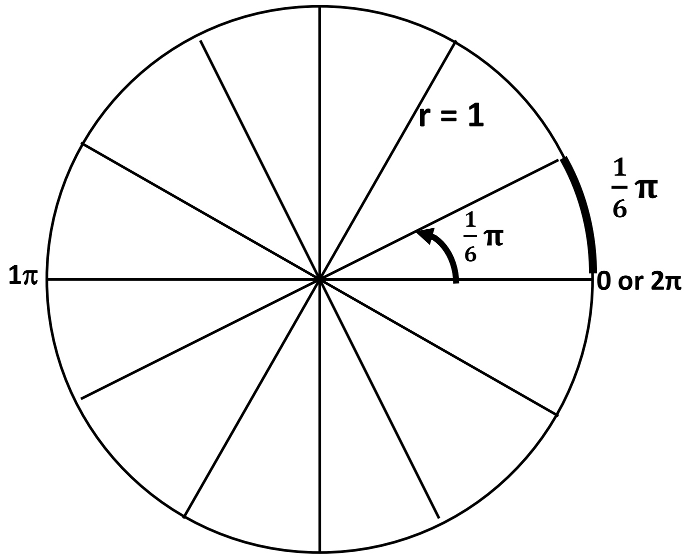
B.

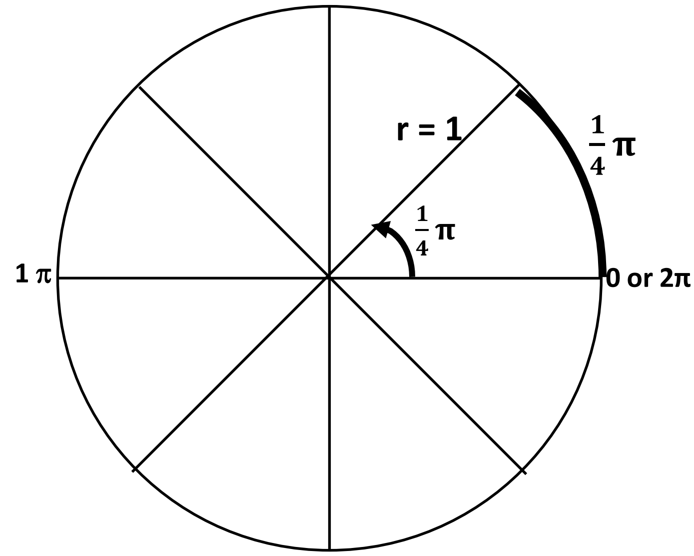


A. Z_1 = _______________ = ___________________ B. Z_2 = _______________ = ___________________

ANGLE MEASURE IN RADIANS All calculations of complex numbers in polar or Euler forms are done in **RADIANS.**

5. Label these "special angles" in Radian Measure (the is, in terms of π). On a Unit Circle, *arc-length = angle*.





Calculations in Euler Form *"Just follow the rules of Real Number Algebra!"* [eg: $(3x^5)(7x^4)$ = _______]

Multiplication: $z_1 * z_2 = r_1\, e^{i\theta_1} * r_2\, e^{i\theta_2} =$ _________ **Division:** $\frac{z_1}{z_2} = \frac{r_1\, e^{i\theta_1}}{r_2\, e^{i\theta_2}} =$ ______________

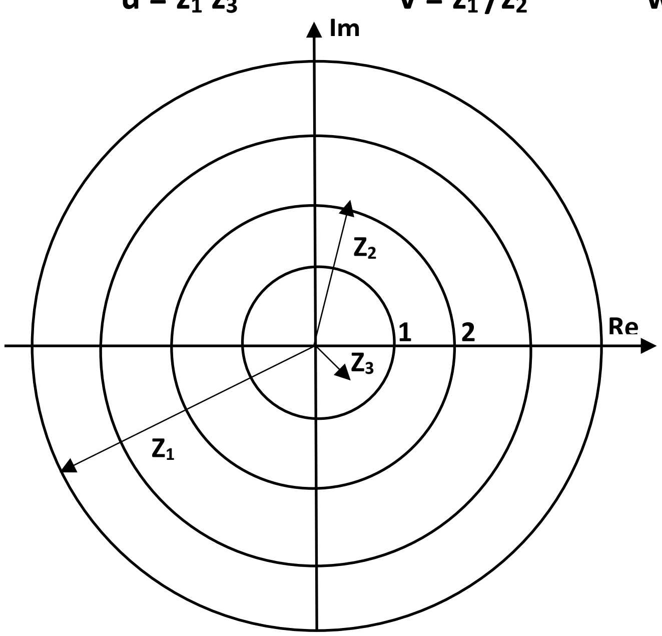
Translate:

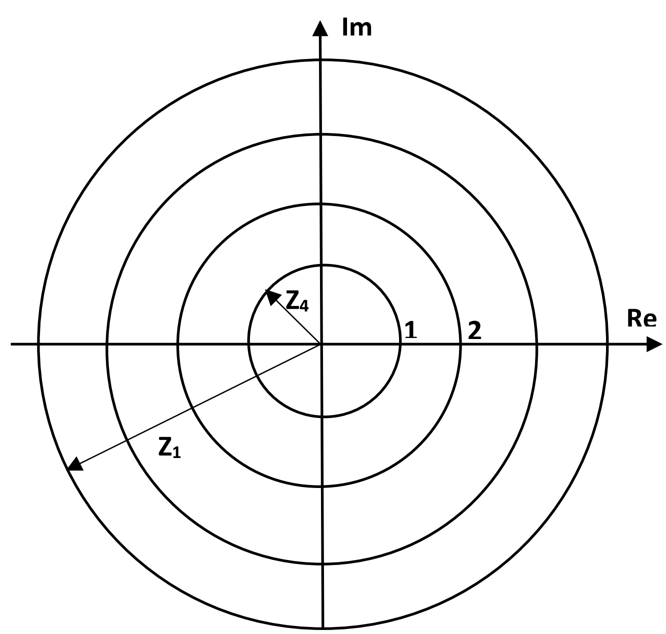
Exponentiation: $z^n = \left(r\, e^{i\theta}\right)^n =$ __________ **Square root:** $\sqrt{z} = \sqrt{(r\, e^{i\theta})}$ = __________________

Translate:

6. Using the given complex numbers, **DRAW** and label each of these complex numbers. These are NOT Special Angles.

$u = z_1 z_3$ **$v = z_1 / z_2$** **$w = z_2^2$** **$x = \sqrt{z_1}$** **$y = z_4^3$**





Challenges: A. Draw the second solution to $x^2 = z_1$

B. If z_4 does equal $e^{i\,3\pi/4}$, then an "unusual" observation from **y** is that: $\mathbf{z_4^3 = z_4}$??

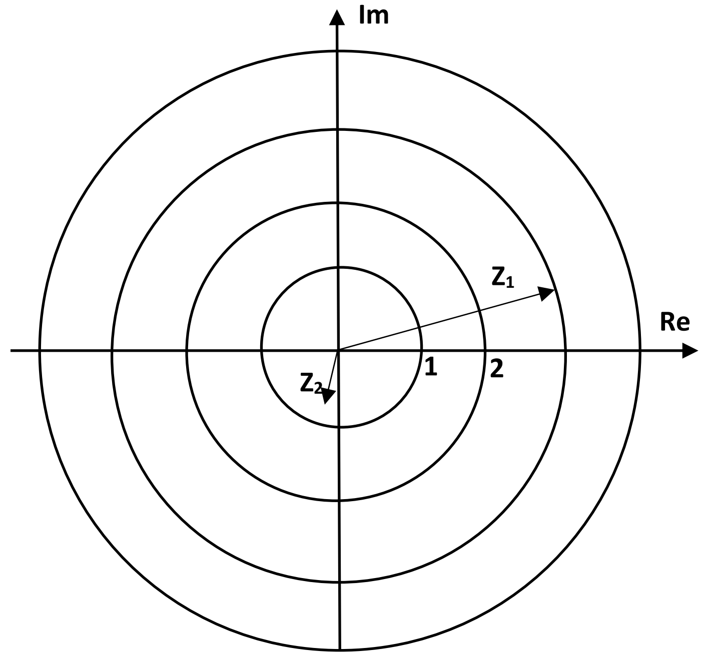
PROBLEM SET

1A. On an Argand plane, locate the complex number ***i***. Use the GEOMETRY to compute:

$i^0 =$ ___; $i^1 =$ ___; $i^2 =$ _____; $i^3 =$ ___; $i^4 =$ ___; ……. $i^{40} =$ ___; … $i^{59} =$ ___;

1B. Using the above pattern, compute:

$i^{13} + i^{32} - i^{35} - i^{50} =$; $i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + \dots + i^{49} + i^{50} =$; $i^1 * i^2 * i^3 * i^4 * i^5 * i^6 * \dots * i^{49} * i^{50} =$



2. If **z = a + bi = r e^{iθ}**,

   then $\bar{z}$ = a – bi = ___________ (Euler)

   and z^{-1} = ________ (Euler)

2A. For each z, DRAW $\bar{z}$ and z^{-1}

2B. Describe all z such that $\bar{z} = z^{-1}$

<u>THEOREM</u>: $\bar{z} = z^{-1}$ if and only if __________?

3. 1957 AMC 12, #42 If $S = i^n + i^{-n}$ and n is an integer, what is the total number of distinct values for S?

4. 1964 AMC 12, #34. If n is a multiple of 4, compute the sum: $1 + 2i + 3i^2 + 4i^4 + 5i^5 + \cdots + (n+1)i^n$.

5. Use the geometry and/or Euler form to simplify:

$\left(\frac{1+i\sqrt{3}}{2}\right)^3 =$ $(1+i)^6 =$

6. 1965 AMC 12, #11. How many and which of these three expressions are incorrect?

$\sqrt{-4} * \sqrt{-16} = \sqrt{(-4)(-16)}$ $\quad\quad$ $\sqrt{(-4)(-16)} = \sqrt{64}$ $\quad\quad$ $\sqrt{64} = 8$

7. Compute: $\left(\frac{1+i\sqrt{3}}{2}\right)^{60} + \left(\frac{1-i\sqrt{3}}{2}\right)^{60} - \frac{(1+i)^5}{(1-i)^3}$

8. Simplify: $\left(\frac{4}{i\sqrt{3}-1}\right)^{12}$

9. 1972 AMC 12, #3 If x = $\frac{1-i\sqrt{3}}{2}$, compute $\frac{1}{x^2-x}$.

[Try both algebraically and geometrically.]

10. 2004 MAθ. If f(z) = z^2 + 4*i* z – 4 and g(z) = $\overline{z}$, what is the value of f(g(3 + 2 *i*)) ?

Hint: Look for short-cuts.

11. 2004 MAθ. If a, b, and $(a + bi)^3$ are non-zero real numbers, compute $\left|\frac{b}{a}\right|$.

12. 1985 AMC 12, #23 If $x = \frac{-1+i\sqrt{3}}{2}$ and $y = \frac{-1-i\sqrt{3}}{2}$, which of the following are incorrect?

$x^5 + y^5 = -1$; $\quad$ $x^7 + y^7 = -1$; $\quad$ $x^9 + y^9 = -1$; $\quad$ $x^{11} + y^{11} = -1$; $\quad$ $x^{13} + y^{13} = -1$’

13. 2008 AIME II, #9 A particle is located on the coordinate plans at (5, 0). Define a *move* for the particle as a counterclockwise rotation of π/4 radians about the origin followed by a translation of 10 units in the positive x direction. After 150 moves, the particle’s position is (p, q).

Determine the greatest integer less than or equal to |p| + |q|.

EXERCISE ANSWERS

1. Solve: $x^2 = 25$, $\mathbf{x = 5}$ **OR** $\mathbf{-5}$; $x^2 = 7$, $\mathbf{x = \pm\sqrt{7}}$; $x^2 = -9$, $x = \mathbf{\pm 3\,i}$; $x^2 + x + 1 = 0$; $x = \frac{-1 \pm i\sqrt{3}}{2}$

2. Simplify: $(7 - 6i) + (5 + i)$; $(-8 + i) - (-3 + 4i)$; $(7 - 6i)(5 + i)$; $(5 + 5i)^2$; $\frac{5-2i}{4+5i} * \frac{4-5i}{4-5i}$

12 – 5i ***–5 – 3i*** ***41 – 23i*** ***– 10i*** $\frac{10\ -33i}{41}$

3. **Theorems**: $\mathbf{z \cdot \bar{z} = a^2 + b^2 = r^2 = |\,z\,|^2}$ **and** $\frac{z}{\bar{z}} = \frac{z}{\bar{z}} * \frac{z}{z} = \frac{z^2}{|z|^2}$ *and later* $= \frac{r^2 e^{i\,2\theta}}{r^2} = e^{i\,2\theta}$

4A. $Z_1 = \sqrt{3} + i = 2\,e^{i\,\pi/6}$ B. $Z_2 = (-3\,,\,3) = 3\sqrt{2}\,e^{i\,3\pi/4}$

Calculations in Euler Form

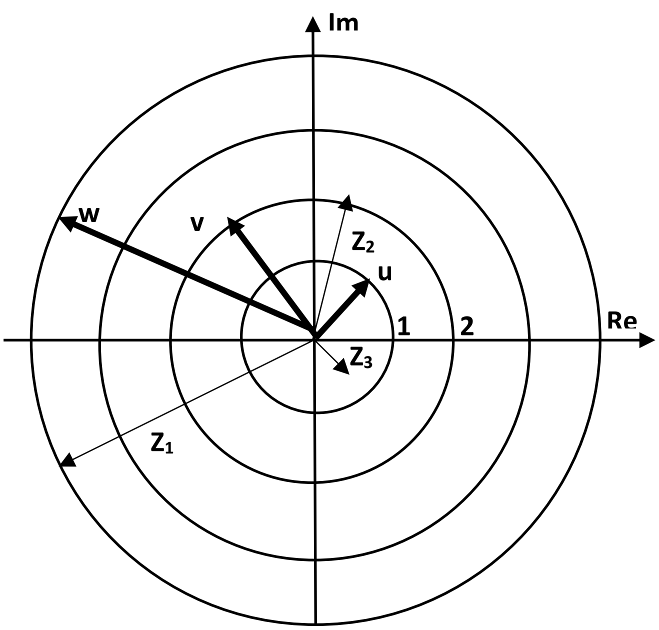
Multiplication: $z_1 * z_2 = r_1\,e^{i\,\theta_1} * r_2\,e^{i\,\theta_2} = r_1 r_2\,e^{i\,(\theta_1+\theta_2)}$ *Multiply the r's; add the angles*

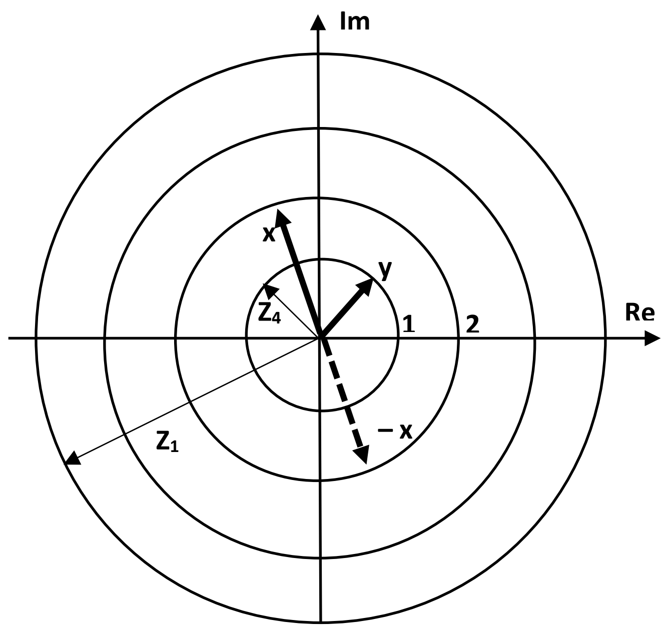
Division: $\frac{z_1}{z_2} = \frac{r_1\,e^{i\,\theta_1}}{r_2\,e^{i\,\theta_2}} = \frac{r_1}{r_2}\,e^{i\,(\theta_1-\theta_2)}$ *Divide the r's; subtract the angles*

Exponentiation: $z^n = \left(r\,e^{i\,\theta}\right)^n = r^n\,e^{i\,n\theta}$ *r^n, then multiply the exponents*

Square root: $\sqrt{z} = \sqrt{(r\,e^{i\,\theta})} = \sqrt{r}\ e^{i\,\theta/2}$ *Same as exponentiation*

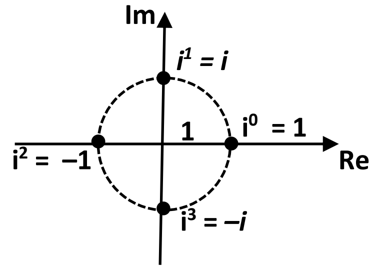
6. $\mathbf{u = z_1\,z_3}$ $\mathbf{v = z_1 / z_2}$ $\mathbf{w = z_2^{\,2}}$ $\mathbf{x = \sqrt{z_1}}$ $\mathbf{y = z_4^{\,3}}$





Challenge: B. If z_4 does equal $e^{i\,3\pi/4}$, then an "unusual" observation is that: $\mathbf{z_4^{\,3} = y = \sqrt[3]{z_4} = z_4^{\,1/3}}$

SOLUTIONS AND HINTS



1A. **1; *i* ; −1 ; −*i* ;** 1; *i* ; −1 ; −*i* ; ETC, a sequence of length 4

$i^{40} = i^0 = 1$; $i^{59} = i^3 = -i$

1B. Using the above pattern, compute:

$i^{13} + i^{32} - i^{35} - i^{50} =$ *i + 1 + i + 1* **= 2 + 2*i***

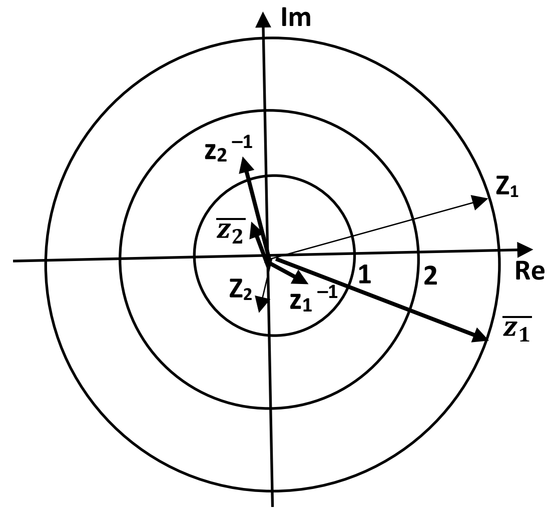
$\underline{i^1 + i^2 + i^3 + i^4} + i^5 + i^6 + ... + i^{49} + i^{50} = i^{49} + i^{50} = i^1 + i^2 =$ **−1 + *i*** because $\underline{i^1 + i^2 + i^3 + i^4} = i - 1 - i + 1 =$ ***0***

$i^1 * i^2 * i^3 * i^4 * i^5 * i^6 * .... * i^{49} * i^{50} = i^{1+2+3...+50} = i^{50*51/2} = i^{1275} = i^3 = \mathbf{-1}$

2. If **z = a + bi = r e^{iθ}**, then $\bar{z}$ **= a − bi** $= re^{-\theta i}$ and $z^{-1} = \left(re^{\theta i}\right)^{-1} = \frac{1}{r}e^{-\theta i}$

Note that $\bar{z}$ and z^{-1} always have the <u>same angle</u>, $-\boldsymbol{\theta}$

2A. For each z, DRAW $\bar{z}$ and z^{-1}



2B. **<u>THEOREM</u>: $\bar{z} = z^{-1}$ if and only if |z| = r = 1**

3. **3** If $S = i^n + i^{-n}$ and n is an integer, what is the total number of distinct values for S?

Testing n = 0,…3, S can equal only −2, 0, or 2 for **<u>three</u>** possibilities.

4. **(2 + n − n*i*)/2** (or equivalent) If n is a multiple of 4, compute:

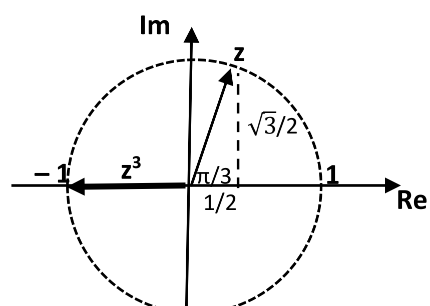
$1 + 2i + 3i^2 + 4i^4 + 5i^5 + \cdots + (n+1)i^n$.

Since n is a multiple of 4 and there are n+1 terms, I will treat the first term "1" as the 'extra' term.

S = 1 + [− 3 + 5 − 7 + 9 − 11… − (n − 1) + (n + 1)] + *i* [2 − 4 + 6 − 8 +…+ (n − 2) − n]

In each bracket, there are exactly n/4 pairs of terms, each pair summing to 2 or −2.

= 1 + 2*n/4 + *i* * (−2)*n/4 **= 1 + n/2 − *i* n/2** = **(2 + n − n*i*)/2**



<u>5A</u>. Note the 30-60-90 triangle.

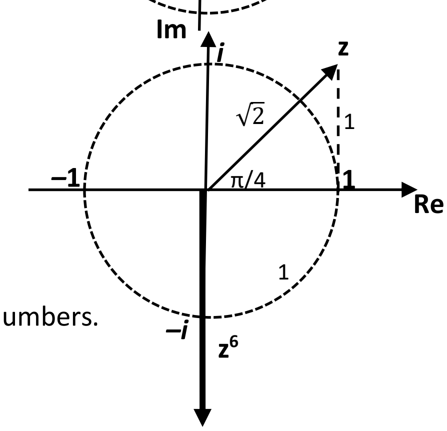
Multiply the angle (60°) by 3. Since r = 1, the *modulus* stays the same.

OR $z^3 = \left(\frac{1+i\sqrt{3}}{2}\right)^3 = \left(e^{i\pi/3}\right)^3 = e^{i\pi} = -\boldsymbol{i}$

<u>B.</u> Note the 45-45-90 triangle with hypotenuse $\sqrt{2}$.

Multiply the angle by 6 and raise r to 6th power**.**

$(1+i)^6 = \left(\sqrt{2}e^{i\pi/4}\right)^6 = -\mathbf{8}\boldsymbol{i}$



<u>6.</u> $\sqrt{-4} * \sqrt{-16} = \sqrt{(-4)(-16)}$; $\sqrt{(-4)(-16)} = \sqrt{64}$; $\sqrt{64} = 8$

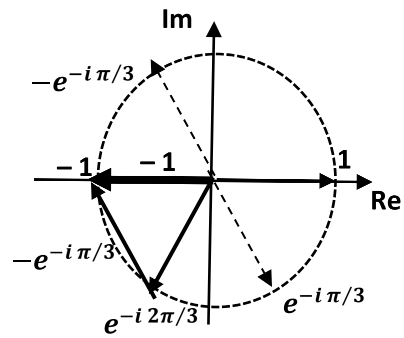
<u>Only the first expression is incorrect</u>. $\sqrt{-4} * \sqrt{-16} = 2i * 4i = -8$, not 8.

<u>NOTE:</u> This means that $\sqrt{x} * \sqrt{y} = \sqrt{xy}$ is <u>NOT</u> a property over the Complex numbers.

7. **0** $\left(\frac{1+i\sqrt{3}}{2}\right)^{60} + \left(\frac{1-i\sqrt{3}}{2}\right)^{60} - \frac{(1+i)^5}{(1-i)^3} = \left(e^{i\pi/3}\right)^{60} + \left(e^{-i\pi/3}\right)^{60} - \frac{\left(\sqrt{2}e^{i\pi/4}\right)^5}{\left(\sqrt{2}e^{-i\pi/4}\right)^3} =$

$1 + 1 - \frac{\sqrt{2}^5\ e^{i\,5\pi/4}}{\sqrt{2}^3\ e^{-i\,3\pi/4}} = 2 - \sqrt{2}^2\ \ e^{8\pi/4} = 2 - 2 = \mathbf{0}$

8. $\left(\frac{4}{i\sqrt{3}-1} * \frac{1}{i\sqrt{3}+1}\right)^{12} = \left(\frac{4}{i\sqrt{3}-1} * \frac{i\sqrt{3}+1}{i\sqrt{3}+1}\right)^{12} = \left(\frac{4(i\sqrt{3}+1)}{-4}\right)^{12} = 2^{12}\left(\frac{(i\sqrt{3}+1)}{2}\right)^{12} = 2^{12}\left(e^{i\pi/3}\right)^{12}$ = **4098**



9. **−1** If x = $\frac{1-i\sqrt{3}}{2} = e^{-i\pi/3}$; $\frac{1}{x^2-x} = \frac{1}{e^{-i\,2\pi/3} - e^{-i\pi/3}}$

Geometry: $e^{-i\,2\pi/3} - e^{-i\pi/3} = -1$; then $\frac{1}{e^{-i\,2\pi/3} - e^{-i\pi/3}} = \frac{1}{-1} = \mathbf{-1}$

Algebra: $x = \frac{1-i\sqrt{3}}{2}$; $x - 1 = \frac{-1-i\sqrt{3}}{2}$; $\frac{1}{x^2-x} = \frac{1}{x(x-1)} = \frac{1}{\frac{1-i\sqrt{3}}{2} * -\frac{1+i\sqrt{3}}{2}} = \frac{-4}{4} = \mathbf{-1}$

10. **9** If f(z) = $z^2 + 4iz - 4$ and g(z) = $\overline{z}$, what is the value of f(g(3 + 2 *i*)) ?

f(z) = $z^2 + 4iz - 4 = (z + 2i)^2$; f(g(3 + 2 *i*)) = f(3 – 2 *i*) = $(3 - 2i + 2i)^2 = 3^2 =$ **9**

11. $\sqrt{3}$ If a, b, and $(a + bi)^3$ are real numbers, compute $\left|\frac{b}{a}\right|$. Let z = a + bi = $re^{i\theta}$. z^3 is a real number only if

θ = π/3 + nπ or θ = 2π/3 + nπ for integral n. For all such values: a = ± 1/2 and b = ±$\sqrt{3}$/2 $\left|\frac{b}{a}\right| = \sqrt{3}$

12. $\mathbf{x^9 + y^9 = -1}$ If $x = \frac{-1+i\sqrt{3}}{2}$ and $y = \frac{-1-i\sqrt{3}}{2}$, which of the following are incorrect?

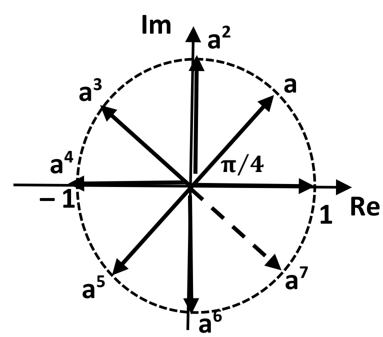
$\mathbf{x^5 + y^5 = -1}$; $\mathbf{x^7 + y^7 = -1}$; $\mathbf{x^9 + y^9 = -1}$; $\mathbf{x^{11} + y^{11} = -1}$; $\mathbf{x^{13} + y^{13} = -1}$

$x = \frac{-1+i\sqrt{3}}{2} = e^{-i\pi/3}$ and $y = \frac{-1-i\sqrt{3}}{2} = e^{i\,2\pi/3}$

$x^9 + y^9 = \left(e^{-i\pi/3}\right)^9 + \left(e^{i\,2\pi/3}\right)^9 = 1 + 1 = 2 \neq -1$ All others do equal −1 .

13. **19** A particle is located on the coordinate plans at (5, 0). Define a move for the particle as a counterclockwise rotation of π/4 radians about the origin followed by a translation of 10 units in the positive x direction. After 150 moves, the particle's position is (p, q). Determine the greatest integer less than or equal to |p| + |q|.

Let *a* = $e^{\pi/4}$ so that multiplication by *a* is a *rotation of π/4 radians clockwise* about the origin. Also, $a^8 = a^{16} = a^{24} = \ldots = 1$



- After 1 move: **5a + 10**; after 2 moves: **a(5a + 10) + 10**;
- After 3 moves: **a(a(5a + 10) + 10) + 10;** After 150 moves:
- $\mathbf{a(\ldots a(a(5a + 10) + 10) + 10\ldots + 10) + 10 = 5a^{150} + 10a^{149} + 10a^{148} + \ldots + 10a + 10}$
- $\mathbf{= 5a^6 + 10(a^{149} + a^{148} + \ldots + a + 1).}$
- There are 150 terms within the parentheses. Each sequence of 8 terms equals 0 [see diagram]. So the first 144 terms "zero out". Also, "opposites" $a^5 + a = 0$ and $a^4 + 1 = 0$.
- = $\mathbf{5a^6 + 10(a^5 + a^4 + a^3 + a^2 + a + 1) = -5i + 10[a^3 + a^2]}$ = $-5i + 10\left[\frac{-\sqrt{2}-i\sqrt{2}}{2} + i\right]$
- = $\mathbf{-5\sqrt{2} + [5 + 5\sqrt{2}]\ i}$ = p + q*i*. So |p| + |q| = | $-5\sqrt{2}$ | + | $5 + 5\sqrt{2}$ | > 7 + 5 + 7 = **19**

Note: This motion creates an octagon!