MO-ARML --- January, 2019 --- GEOMETRY OF COMPLEX NUMBERS

Many Complex Number problems are most easily solved geometrically on the complex Argand Plane.

GOAL: Treat these as 10-second Mental Arithmetic (or geometry) problems: Simplify \( \left( \frac{1 + i \sqrt{3}}{2} \right)^3 \) and \( (1 + i)^6 \)

First, the BASICS

1. Solve: \( x^2 = 25 \), \( x = \phantom{0} \); \( x^2 = 7 \), \( x = \phantom{0} \); \( x^2 = -9 \), \( x = \phantom{0} \); \( x^2 + x + 1 = 0 \); \( x = \phantom{0} \)

2. Simplify: \((7 - 6i) + (5 + i)\); \((-8 + i) - (-3 + 4i)\); \((7 - 6i)(5 + i)\); \((5 + 5i)^2\); \(\frac{5 - 2i}{4 + 5i}\)

3. If \( z = a + bi \), then the conjugate of \( z \) is \( \bar{z} = a - bi \).

**Theorems:** \( Z \cdot \bar{Z} = \phantom{0} = \phantom{0} = \phantom{0} \) and \( \frac{z}{\bar{z}} = \phantom{0} \)

**Argand Diagram** Any Real Number can be represented as a point on a Real Number Line [one-dimensional]. Any Complex Number can be represented as a point on the Complex Plane [two-dimensional].

4. Write each complex number in both rectangular and Euler form.

**Hints:**

A. \( \sqrt{3} \) Re

\( Z_1 = \phantom{0} \) = \phantom{0} \)

B. \( \sqrt{2} \) Re

\( Z_2 = \phantom{0} \) = \phantom{0} \)
**ANGLE MEASURE IN RADIANS**  All calculations of complex numbers in polar or Euler forms are done in **RADIANS**.

5. Label these “special angles” in Radian Measure (the is, in terms of π). On a **Unit Circle**, 
   arc-length = angle.

6. Using the given complex numbers, **DRAW** and label each of these complex numbers. These are NOT Special Angles.

   \[ u = z_1 \overline{z_3} \quad v = z_1 / z_2 \quad w = z_2^2 \quad x = \sqrt{z_1} \quad y = z_4^3 \]

**Challenges:**

A. Draw the second solution to \( x^2 = z_1 \)

B. If \( z_4 \) does equal \( e^{i \cdot \pi/4} \), then an “unusual” observation from \( y \) is that: \( z_4^3 = z_4 ?? \)
PROBLEM SET

1A. On an Argand plane, locate the complex number $i$. Use the GEOMETRY to compute:

\[ i^0 = __; \quad i^1 = __; \quad i^2 = __; \quad i^3 = __; \quad i^4 = __; \quad \ldots \quad i^{10} = __; \quad \ldots \quad i^{59} = __; \]

1B. Using the above pattern, compute:

\[ i^{13} + i^2 - i^5 - i^{50} = ; \quad i^1 + i^2 + i^3 + i^4 + i^5 + \ldots + i^{49} + i^{50} = ; \quad i^1 \cdot i^2 \cdot i^3 \cdot i^4 \cdot i^5 \cdot \ldots \cdot i^{49} \cdot i^{50} = \]

2. If $z = a + bi = r e^{i\theta}$,

then $\bar{z} = a - bi = \underline{\text{____________}}$ (Euler)

and $z^{-1} = \underline{\text{_________}}$ (Euler)

2A. For each $z$, DRAW $\bar{z}$ and $z^{-1}$

2B. Describe all $z$ such that $\bar{z} = z^{-1}$

THEOREM: $\bar{z} = z^{-1}$ if and only if _________?

3. 1957 AMC 12, #42 If $S = i^n + i^{-n}$ and $n$ is an integer, what is the total number of distinct values for $S$?

4. 1964 AMC 12, #34. If $n$ is a multiple of 4, compute the sum: $1 + 2i + 3i^2 + 4i^4 + 5i^5 + \cdots + (n + 1)i^n$.

5. Use the geometry and/or Euler form to simplify:

\[ \left( \frac{1+i\sqrt{3}}{2} \right)^3 = \quad (1 + i)^6 = \]
6. 1965 AMC 12, #11. How many and which of these three expressions are incorrect?
\[
\sqrt{-4} \cdot \sqrt{-16} = \sqrt{(-4)(-16)} \quad \sqrt{(-4)(-16)} = 64 \quad \sqrt{64} = 8
\]

7. Compute: \[
\left( \frac{1+i\sqrt{3}}{2} \right)^{60} + \left( \frac{1-i\sqrt{3}}{2} \right)^{60} - \frac{(1+i)^5}{(1-i)^3}
\]

8. Simplify: \[
\left( \frac{4}{i\sqrt{3}-1} \right)^{12}
\]

9. 1972 AMC 12, #3 If \( x = \frac{1-i\sqrt{3}}{2} \), compute \( \frac{1}{x^2-x} \).
   [Try both algebraically and geometrically.]

10. 2004 MA0. If \( f(z) = z^3 + 4i z - 4 \) and \( g(z) = \overline{z} \), what is the value of \( f(g(3 + 2i)) \)?
    Hint: Look for short-cuts.

11. 2004 MA0. If \( a, b, \) and \((a + bi)^3\) are non-zero real numbers, compute \( \left| \frac{b}{a} \right| \).

12. 1985 AMC 12, #23 If \( x = \frac{-1+i\sqrt{3}}{2} \) and \( y = \frac{-1-i\sqrt{3}}{2} \), which of the following are incorrect?
    \[
    x^5 + y^5 = -1; \quad x^7 + y^7 = -1; \quad x^9 + y^9 = -1; \quad x^{11} + y^{11} = -1; \quad x^{13} + y^{13} = -1.'
    \]

13. 2008 AIME II, #9 A particle is located on the coordinate plans at (5, 0). Define a move for the particle as a counterclockwise rotation of \( \pi/4 \) radians about the origin followed by a translation of 10 units in the positive x direction. After 150 moves, the particle's position is (p, q).
    Determine the greatest integer less than or equal to |p| + |q|. 
EXERCISE ANSWERS

1. Solve: \( x^2 = 25 \); \( x = 5 \text{ OR } -5 \); \( x^2 = 7 \); \( x = \pm \sqrt{7} \); \( x^2 = -9 \); \( x = \pm 3i \); \( x^2 + x + 1 = 0 \); \( x = \frac{-1 \pm i\sqrt{3}}{2} \)

2. Simplify:
   - \((7 - 6i) + (5 + i)\)
   - \((-8 + i) - (-3 + 4i)\)
   - \((7 - 6i)(5 + i)\)
   - \((5 + 5i)^2\)
   - \(\frac{5 - 2i}{4 + 5i} \cdot \frac{4 - 5i}{4 - 5i}\)
   - \(\frac{10 - 33i}{41}\)

3. Theorems:
   - \(z \cdot \bar{z} = a^2 + b^2 = r^2 = |z|^2\)
   - \(\frac{z}{\bar{z}} = \frac{z}{\bar{z}} = \frac{z^2}{|z|^2}\) and later \(\frac{r^2 e^{i2\theta}}{r^2} = e^{i2\theta}\)

4. A. \(Z_1 = \sqrt{3} + i = 2 e^{i\pi/6}\)
   - B. \(Z_2 = (-3, 3) = 3\sqrt{2} e^{i3\pi/4}\)

Calculations in Euler Form

Multiplication: \(z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}\)

Division: \(\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\)

Exponentiation: \(z^n = (r e^{i\theta})^n = r^n e^{i n\theta}\)

Square root: \(\sqrt{z} = \sqrt{r e^{i\theta}} = \sqrt{r} e^{i \theta/2}\)

6. \(u = z_1 z_3\)
   - \(v = z_1 / z_2\)
   - \(w = z_2^2\)
   - \(x = \sqrt{z_1}\)
   - \(y = z_4^3\)

Challenge: B. If \(z_4\) does equal \(e^{i3\pi/4}\), then an “unusual” observation is that: \(z_4^3 = y = \sqrt[3]{z_4} = z_4^{1/3}\)
SOLUTIONS AND HINTS

1A. \(i; -i; \quad 1; \quad i; \quad -1; \quad -i; \quad 1; \quad i; \quad -1; \quad -i; \quad \text{ETC, a sequence of length 4}
\)
\(p^0 = p = 1; \quad p^2 = p^3 = -i\)

18. Using the above pattern, compute:
\[ i^{13} + i^{32} - i^{53} - i^{20} = i + 1 + i + 1 = 2 + 2i \]
\[ i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + \ldots + i^{49} + i^{50} = i^{49} + i^{50} = i^1 + i^2 = -1 + i \quad \text{because} \quad i^1 + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0 \]
\[ i^1 * i^2 * i^3 * i^5 * i^6 * \ldots * i^{49} * i^{50} = i^{1+2+3+\ldots+50} = i^{1275} = i^3 = -1 \]

2. If \(z = a + bi = re^\theta i\), then \(\bar{z} = a - bi = re^{-\theta}i\) and \(z^{-1} = (re^\theta i)^{-1} = \frac{1}{r}e^{-\theta}i\)

Note that \(\bar{z}\) and \(z^{-1}\) always have the same angle, \(-\theta\)

2A. For each \(z\), DRAW \(\bar{z}\) and \(z^{-1}\)

2B. \textbf{THEOREM:} \(\bar{z} = z^{-1}\) if and only if \(|z| = r = 1\)

3. \(S = i^n + i^{-n}\) and \(n\) is an integer, what is the total number of distinct values for \(S\)?

Testing \(n = 0, 1, 2, 3\), \(S\) can equal only \(-2, 0, 2\) for \(\text{three}\) possibilities.

4. \((2 + n - ni)/2\) (or equivalent) If \(n\) is a multiple of 4, compute:
\[ 1 + 2i + 3i^2 + 4i^3 + 5i^4 + \ldots + (n + 1)i^n . \]

Since \(n\) is a multiple of 4 and there are \(n+1\) terms, I will treat the first term “1” as the ‘extra’ term.
\[ S = 1 + [ -3 + 5 - 7 + 9 - 11 \ldots - (n - 1) + (n + 1)] + i[2 - 4 + 6 - 8 + \ldots + (n - 2) - n] \]

In each bracket, there are \(\frac{n}{2}\) pairs of terms, each pair summing to 2 or \(-2\).
\[ = 1 + 2*\frac{n}{4} + i*(-2)*\frac{n}{4} = 1 + \frac{n}{2} - i\frac{n}{2} = (2 + n - ni)/2 \]

5A. Note the 30-60-90 triangle.

Multiply the angle \((60^\circ)\) by 3. Since \(r = 1\), the \textit{modulus} stays the same.
\[ \text{OR } z^3 = \left( \frac{1+i\sqrt{3}}{2} \right)^3 = \left( e^{i \pi/3} \right)^3 = e^{i \pi} = -i \]

B. Note the 45-45-90 triangle with hypotenuse \(\sqrt{2}\).

Multiply the angle by 6 and raise \(r\) to the \(6^\text{th}\) power.
\[ (1 + i)^6 = \left( \sqrt{2}e^{i \pi/4} \right)^6 = -8i \]

6. \(\sqrt{-4} * \sqrt{-16} = \sqrt{(-4)(-16)} \); \(\sqrt{(-4)(-16)} = \sqrt{64} \); \(\sqrt{64} = 8 \)

\textbf{Only the first expression is incorrect.} \(\sqrt{-4} * \sqrt{-16} = 2i * 4i = -8\), not 8.

\textbf{NOTE:} This means that \(\sqrt{x} * \sqrt{y} = \sqrt{xy}\) is \textbf{NOT} a property over the Complex numbers.
7. \(\left(\frac{1+i}{\sqrt{2}}\right)^{60} + \left(\frac{1-i}{\sqrt{2}}\right)^{60} - \frac{(1+i)^5}{(1-i)^3} = (e^{i \pi/3})^{60} + (e^{-i \pi/3})^{60} - \left(\frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}}\right)^5 = 0 \)

\[
1 + \frac{\sqrt{2}^5 e^{5\pi/4}}{\sqrt{2}^5 e^{-13\pi/4}} = 2 - \sqrt{2}^2 e^{8\pi/4} = 2 - 2 = 0
\]

8. \(\frac{4}{i\sqrt{3} - 1} + \frac{1}{i\sqrt{3} + 1}\)^12 = \(\frac{4}{i\sqrt{3} - 1} \cdot \frac{i\sqrt{3} + 1}{i\sqrt{3} + 1}\)^12 = \(\frac{4(i\sqrt{3} + 1)}{-4}\)^12 = \(2\sqrt{2}\left(\frac{(i\sqrt{3} + 1)}{2}\right)^{12}\) = \(2\sqrt{2}(e^{i\pi/3})^{12}\) = \(4098\)

9. \(-1\) If \(x = \frac{1-i}{\sqrt{2}} = e^{-i\pi/3}\), then \(\frac{1}{x^2 - x} = \frac{1}{e^{-2i\pi/3} - e^{-i\pi/3}}\)

Geometry: \(e^{-2i\pi/3} - e^{-i\pi/3} = -1\); then \(\frac{1}{e^{-2i\pi/3} - e^{-i\pi/3}} = \frac{1}{-1} = -1\)

Algebra: \(x = \frac{1-i}{\sqrt{2}}\); \(x - 1 = \frac{-1-i}{\sqrt{2}}\); \(\frac{1}{x(x-1)} = \frac{1}{x^2-x} = \frac{1}{\frac{-1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}} = \frac{-4}{4} = -1\)

10. \(f(z) = z^2 + 4iz - 4\) and \(g(z) = \overline{z}\), what is the value of \(f(g(3 + 2i))\)?

\[f(z) = z^2 + 4iz - 4 = (z + 2i)^2; \quad f(g(3 + 2i)) = f(3 - 2i) = (3 - 2i)^2 = 3^2 = 9\]

11. \(\sqrt{3}\) If \(a, b, \text{ and } (a + bi)^3\) are real numbers, compute \(\frac{b}{a}\). Let \(z = a + bi = r e^{i \theta}\). \(z^3\) is a real number only if \(\theta = \pi/3 + n\pi\) or \(\theta = 2\pi/3 + n\pi\) for integral \(n\). For all such values: \(a = \pm 1/2\) and \(b = \pm \sqrt{3}/2\) \(\frac{b}{a} = \sqrt{3}\)

12. \(x^8 + y^9 = -1\) If \(x = \frac{-1+i}{\sqrt{2}}\) and \(y = \frac{-1-i}{\sqrt{2}}\), which of the following are incorrect?

\[x^5 + y^5 = -1; \quad x^7 + y^7 = -1; \quad x^9 + y^9 = -1; \quad x^{11} + y^{11} = -1; \quad x^{13} + y^{13} = -1\]

\[x = \frac{-1+i}{\sqrt{2}} = e^{-i\pi/3}\] and \(y = \frac{-1-i}{\sqrt{2}} = e^{i\pi/3}\)

\[x^3 + y^9 = \left(e^{-i\pi/3}\right)^3 + \left(e^{i\pi/3}\right)^9 = 1 + 1 = 2 \neq -1\] All others do equal \(-1\).

13. \(19\) A particle is located on the coordinate plane at \((5, 0)\). Define a move for the particle as a counterclockwise rotation of \(\pi/4\) radians about the origin followed by a translation of 10 units in the positive \(x\) direction. After 150 moves, the particle’s position is \((p, q)\). Determine the greatest integer less than or equal to \(|p| + |q|\).

Let \(a = e^{\pi i/4}\) so that multiplication by \(a\) is a rotation of \(\pi/4\) radians clockwise about the origin. Also, \(a^8 = a^{16} = a^{24} = ... = 1\)

- After 1 move: \(5a + 10\); after 2 moves: \(a(5a + 10) + 10\);
- After 3 moves: \(a(a(5a + 10) + 10) + 10\); After 150 moves:
- \(a(...)a(5a + 10) + 10... + 10 + 10 = 5a^{150} + 10a^{149} + 10a^{148} + ... + 10a + 10\)
- \(= 5a^{6} + 10(a^{49} + a^{48} + ... + a + 1)\).
- There are 150 terms within the parentheses. Each sequence of 8 terms equals 0 [see diagram]. So the first 144 terms “zero out”. Also, “opposites” \(a^5 + a = 0\) and \(a^4 + 1 = 0\).
- \(= 5a^6 + 10(a^2 + a^3 + a^2 + a + 1) = -5i + 10[a^3 + a^2] = -5i + 10 \left[ \frac{-\sqrt{2} - i\sqrt{2}}{2} + i \right]\)
- \(= -5\sqrt{2} + [5 + 5\sqrt{2}]i = p + qi\). So \(|p| + |q| = | -5\sqrt{2} | + | 5 + 5\sqrt{2} | > 7 + 5 = 19\)

Note: This motion creates an octagon!