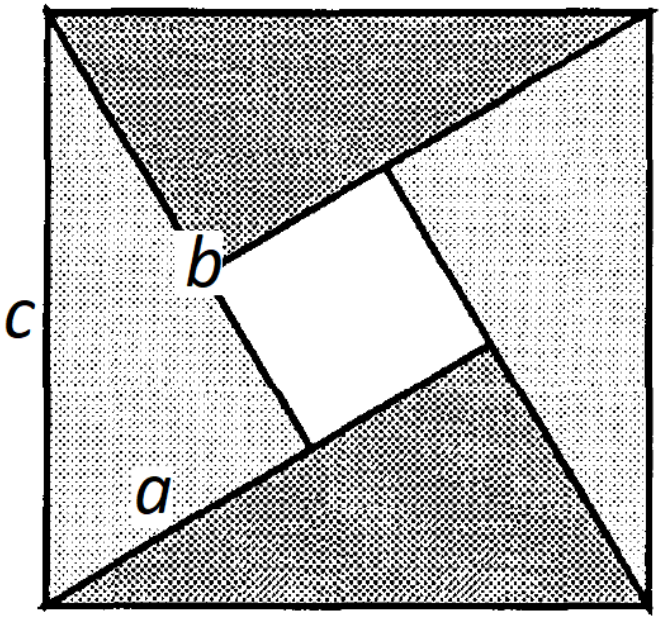
WUSTL Math Circle

Proofs Without Words II

(based on Nelson, R. G., *Proofs Without Words*, MAA, 1993.)

1. What can you deduce from the following figure?



Answer. The large square (of area c^2) is decomposed into four congruent right triangles (each of area $\frac{ab}{2}$) and a small square of side $b-a$ (so of area $(b-a)^2$). Therefore

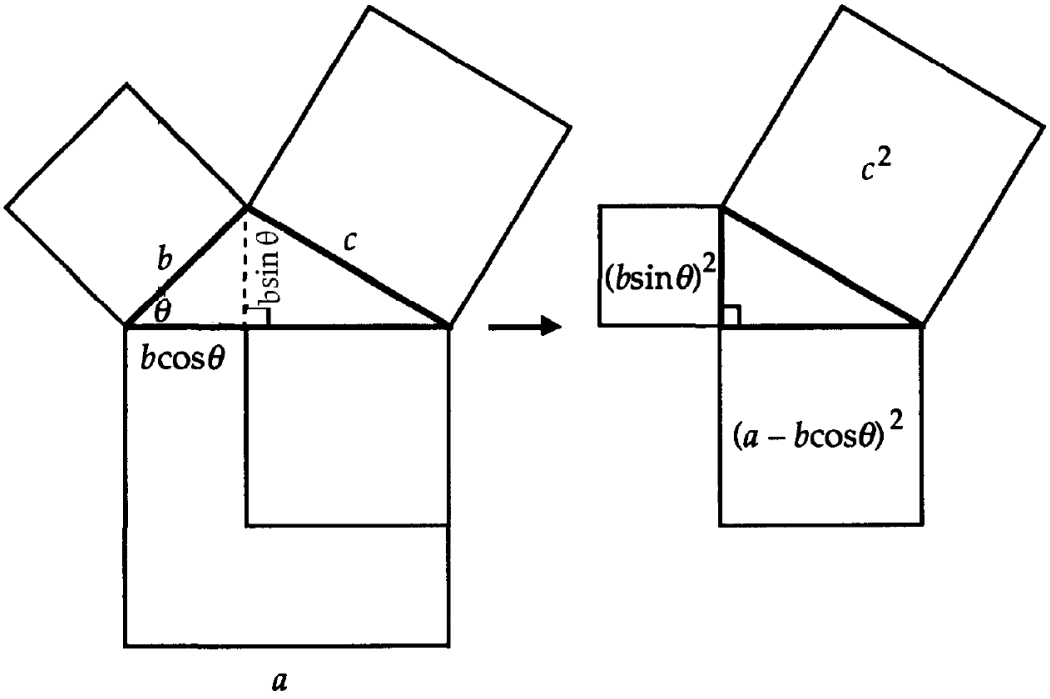
$$c^2 = 4 \times \frac{ab}{2} + (b-a)^2 = 2ab + b^2 - 2ab + a^2 = a^2 + b^2,$$

namely

$$c^2 = a^2 + b^2.$$

This is Pythagoras' Theorem: *in a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides*.

2. Use the following figure as well as the result of the previous activity to derive a formula for c in terms of a, b and θ.



3. Start with angle BOA_0.

   Bisect it by BOA_1.

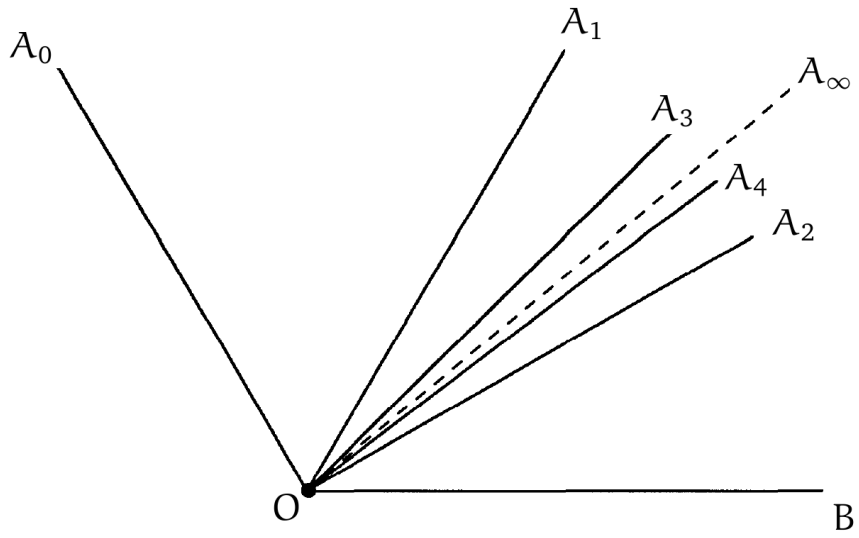
   Bisect the latter by A_1OA_2.

   Bisect the latter by A_2OA_3.

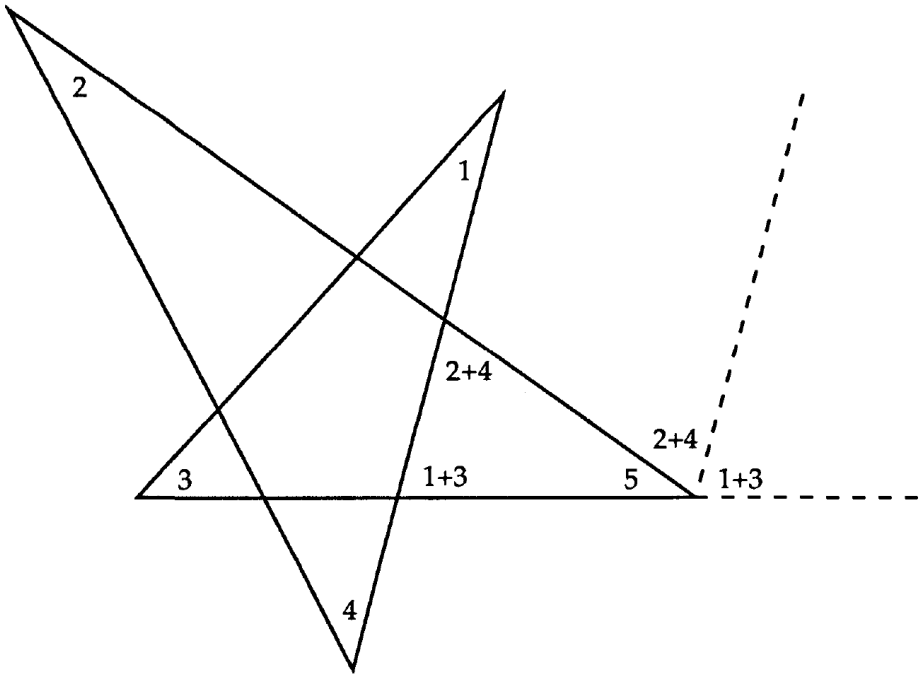
   Bisect the latter by A_3OA_4.

   Continue.

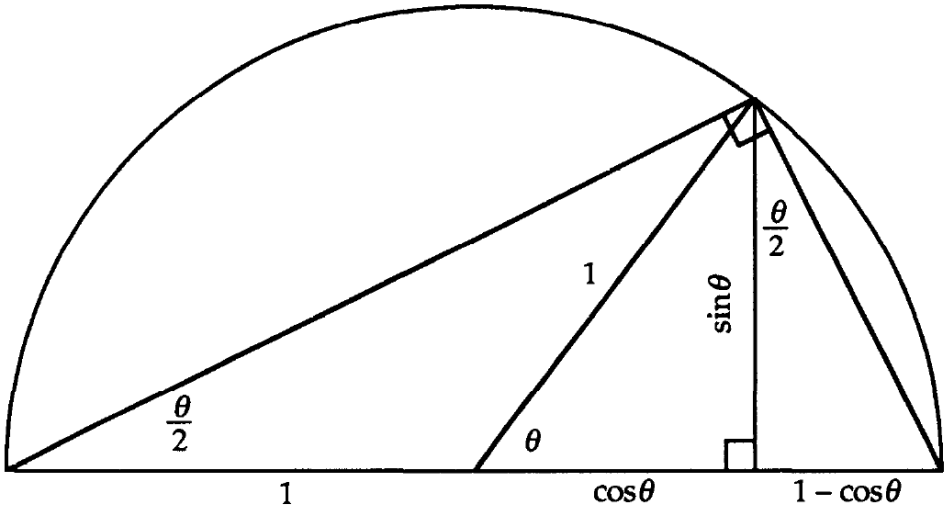
   What is angle BOA_∞ compared to BOA_0?



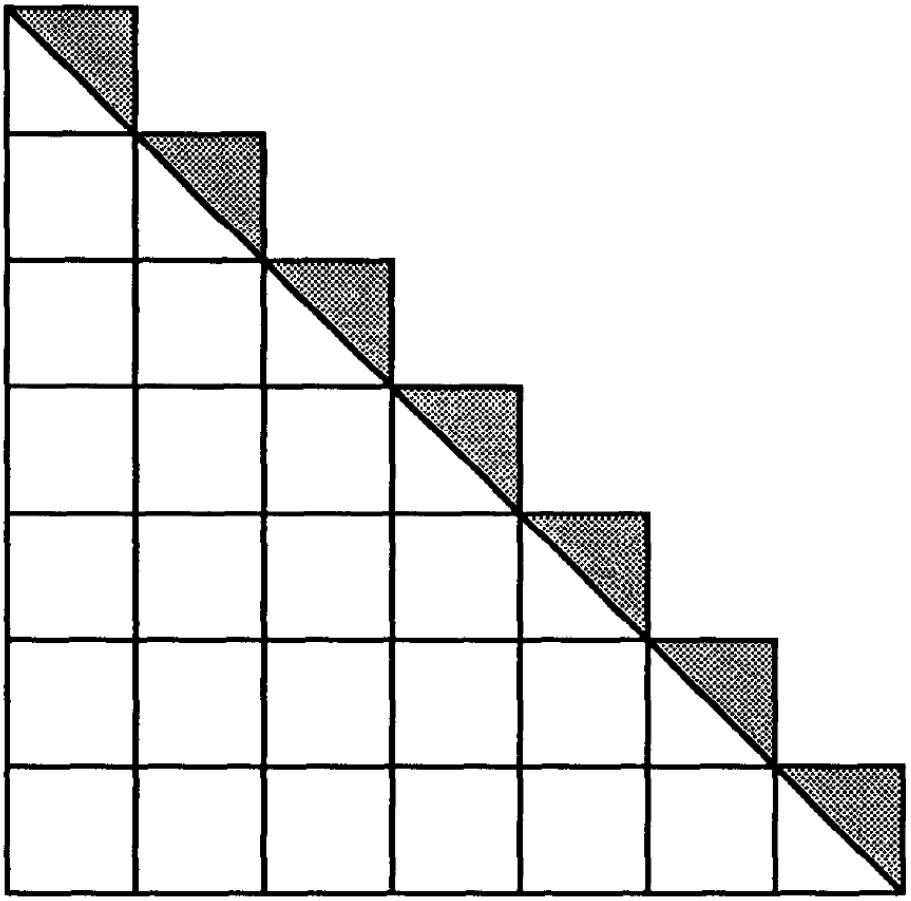
4. What is the sum of the angles of a star?



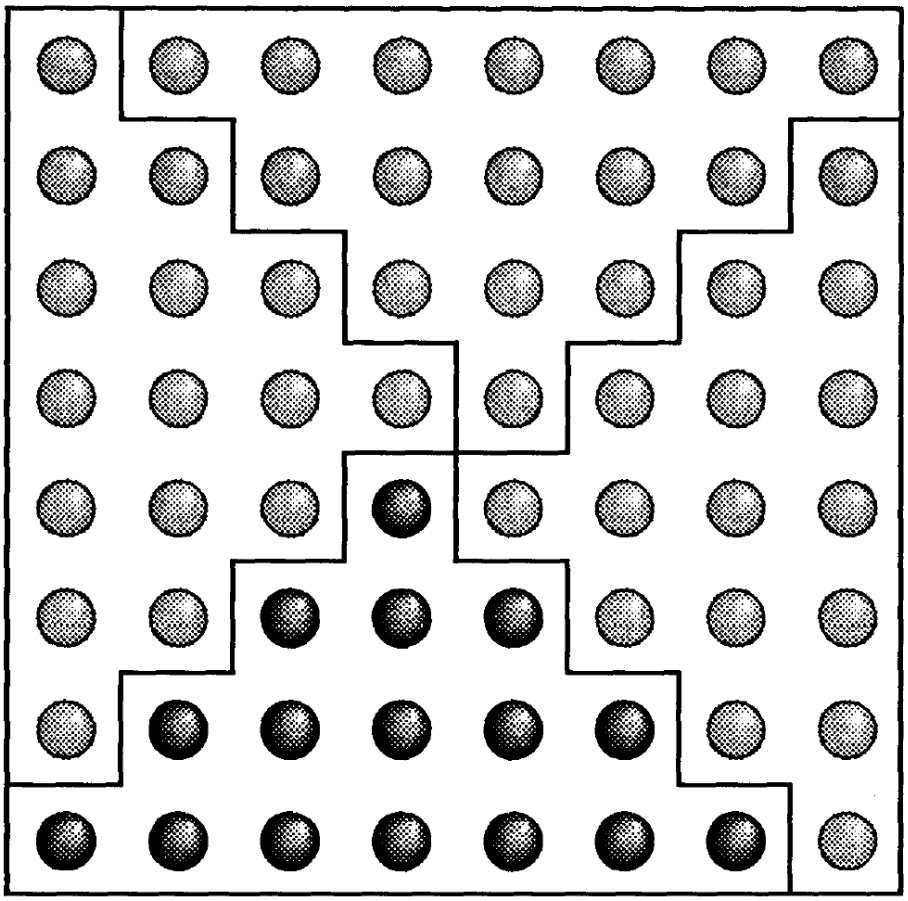
5. Using the following figure derive a formula for $\tan \frac{\theta}{2}$ in terms of $\sin\theta$ and $\cos\theta$.



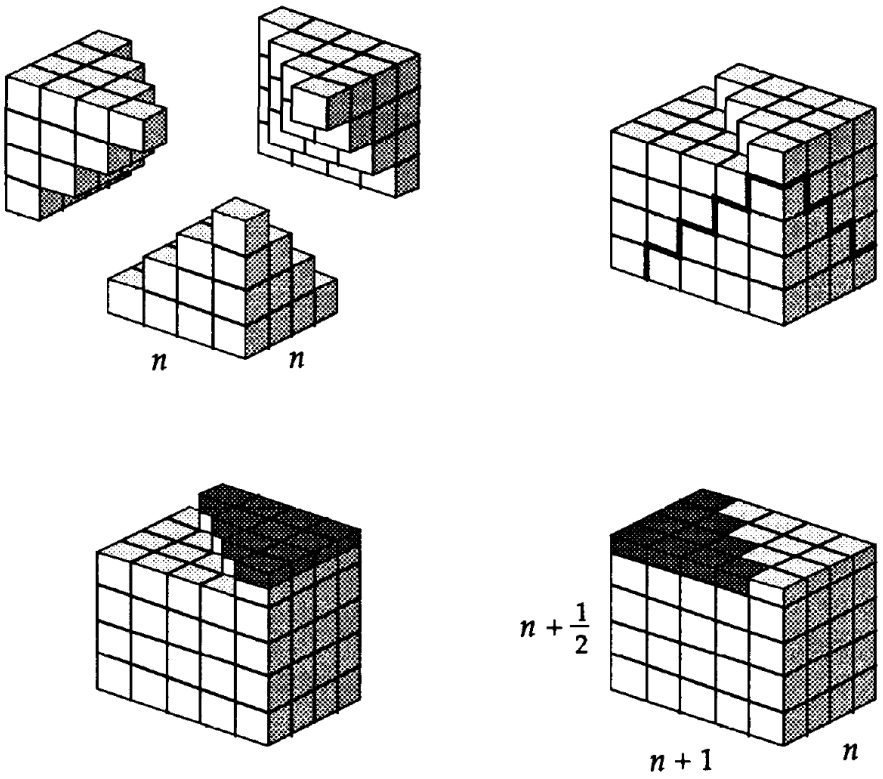
6. Derive a formula for $1 + 2 + \cdots + n$ using the following figure:



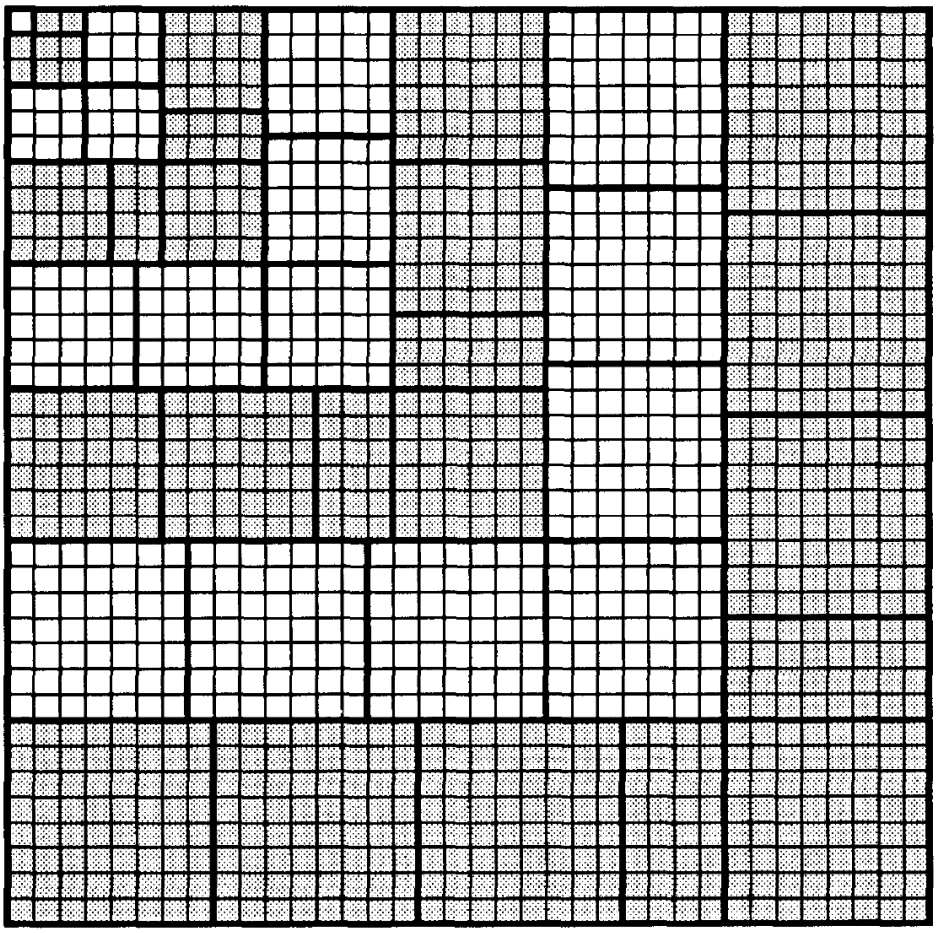
7. Derive a formula for $1 + 3 + 5 + \cdots + (2n - 1)$ using the following figure:



8. Derive a formula for $1^2 + 2^2 + 3^2 + \cdots + n^2$ from the following figure:

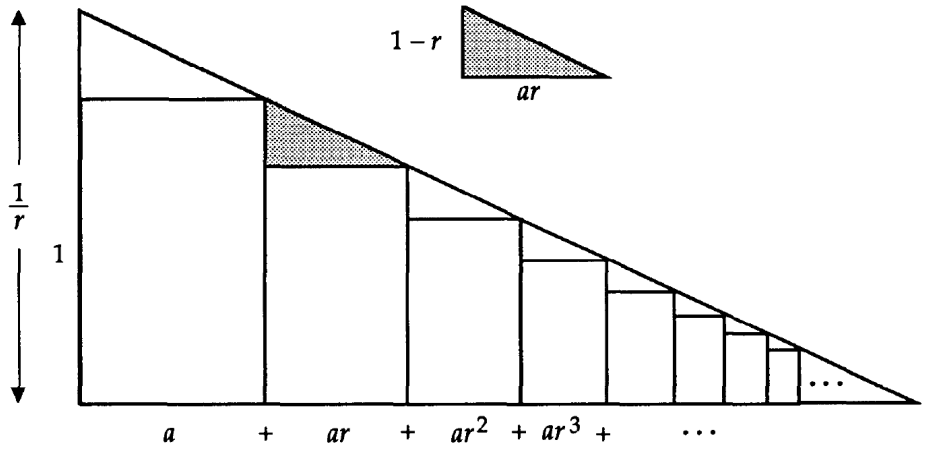


9. Derive a formula for $1^3 + 2^3 + 3^3 + \cdots + n^3$ using the following figure:



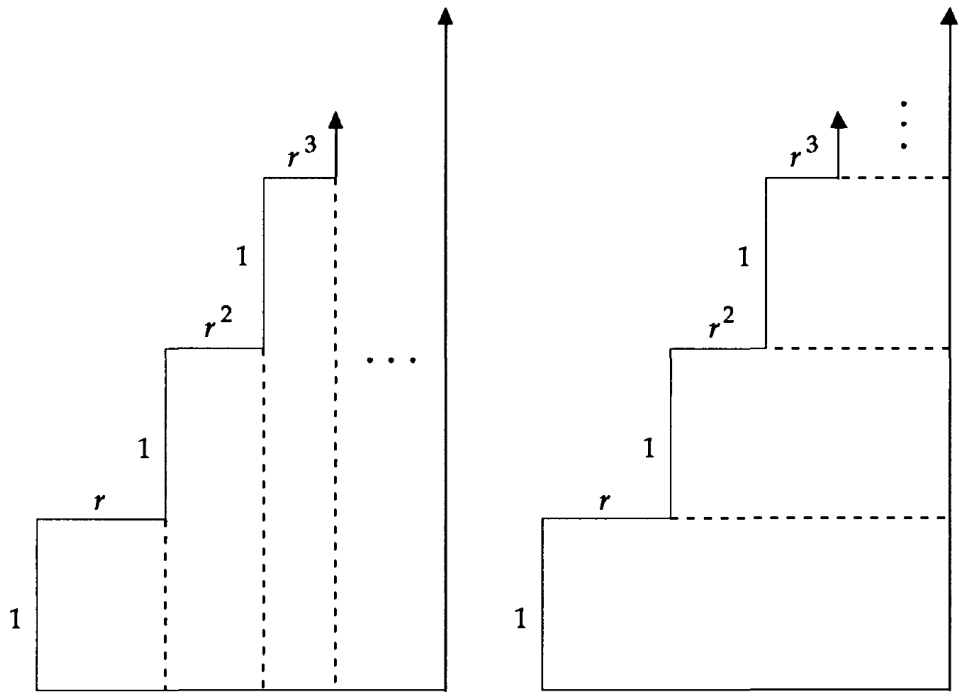
10. Let r be a positive real number less than 1. In the following figure, use the similarity between the distinguished small triangle and the big triangle in order to find a formula for

$$a + ar + ar^2 + ar^3 + \cdots .$$

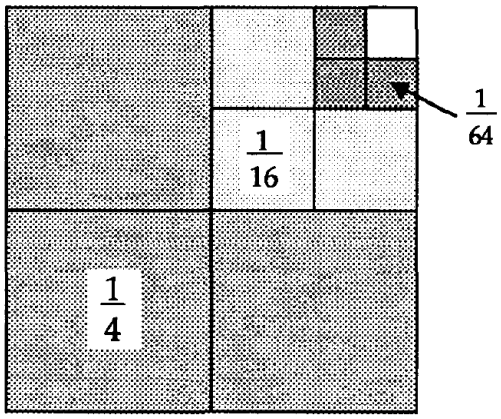


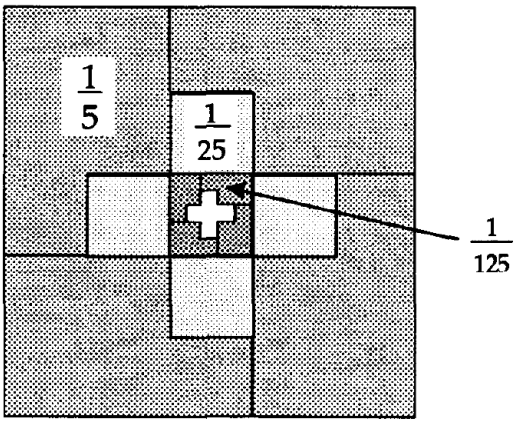
11. Let r be a positive real number less than 1. Use the following figure and the formula you have derived through the previous activity to find a formula for

$$r + 2r^2 + 3r^3 + \cdots .$$

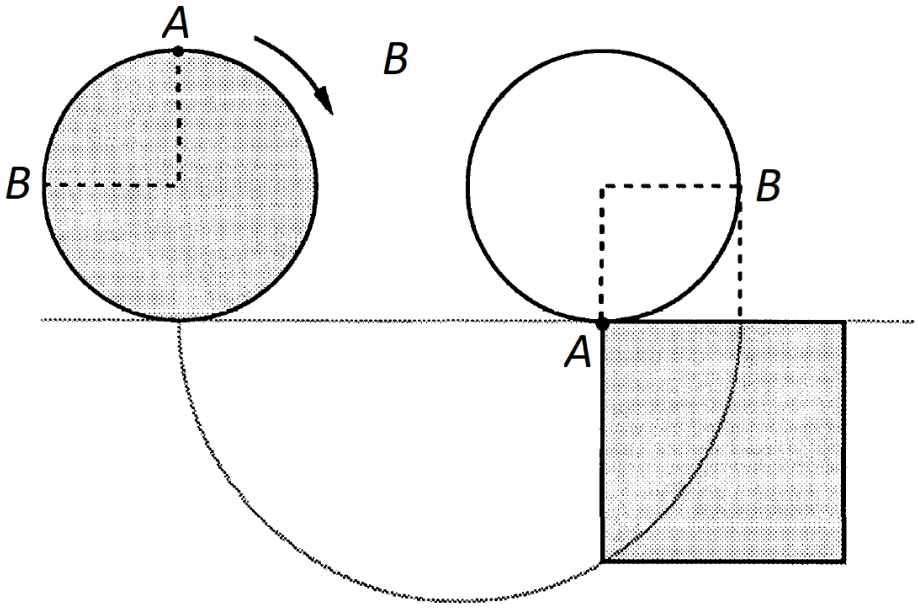


12. Compute $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots$ and $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$ using the following figure:





13. We half rolled a small disk. What is the area of the distinguished square compared to our disk?



14. From an arbitrary point P inside the equilateral triangle ABC we draw perpendiculars PD, PE and PF. Use the following figure to find the sum of lengths $PE + PE + PF$.