# FANTASTIC FACTORING

*[Thanks to my friend Harold Reiter of North Carolina for much of this material.]*

Following are some factoring patterns, formulas, and a theorem that you might already recognize.

**Difference of squares**

$x^2 - y^2 = (x - y)(x + y)$

**Difference of Cubes**

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

**Sum of Cubes**

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

**Pascal's Triangle**

| | | | | | | |
|---|---|---|---|---|---|---|
| **1** | **1** | | | | | |
| **1** | **2** | **1** | | | | |
| **1** | **3** | **3** | **1** | | | |
| **1** | **4** | **6** | **4** | **1** | | |
| **1** | **5** | **10** | **10** | **5** | **1** | |
| **1** | **6** | **15** | **20** | **15** | **6** | **1** |

**Vieta's Formulas** connect the coefficients of a polynomial to sums and products of its roots.

This quartic example suggests Viete's formulas [but be careful to correctly assign plus and minus signs].

$(x - p)(x - q)(x - r)(x - s) = x^4 - (p+q+r+s)\, x^3 + (pq+pr+ps+qr+qs+rs)\, x^2 - (pqr+pqs+prs+qrs)\, x + pqrs$

**THEOREM**: Given function f(x) and constant x = a, the following four statements are equivalent.

**1.** $f(a) = 0$ **2**. $x - a$ is a factor of $f(x)$. **3**. $x = a$ is a zero of $f(x)$ **4.** $(a, 0)$ is an x-intercept of graph of $f(x)$

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Factor the following expressions.

1. $16x^4 - 81y^4$

2. $125x^3 + 64y^3$

3. $8x^3 - 27$

4. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

5. $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

6. There are two distinct ways to "attack" the factorization of $x^6 - y^6$. Try both ways and then convince yourself that they are equivalent.

A. $x^6 - y^6 =$

B. $x^6 - y^6 =$

7. By studying the patterns for *Sum and Difference of Cubes*, above, can you determine how to factor each of these?

A. $x^7 + y^7 =$

B. $x^{10} - y^{10} =$

C. **Explain** why $x^8 + y^8$ cannot be factored in a similar way.

8. One technique for factoring expressions with 4 or more terms is ***factor by grouping.*** Try these.

A. $4ab - 8b^2 + 3a^3 - 6a^2b$

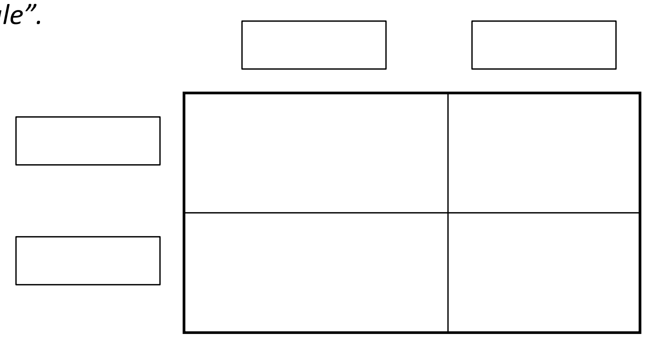
B. $xy + x + y + 1$

C. Can you expand $(x + 1)(y + 1)(z + 1)$ in "one step"?

To solve most polynomial equations, you set an expression equal to zero and factor it. However, if you are told that the solutions are ***integers,*** other methods are possible.

9. If x is a positive integer, solve: $x(x + 1)(x + 2)(x + 3) + 1 = 379^2$

***Simon's Favorite Factoring Trick (SFFT***) is a great tool for solving certain math contest problems. I will present this example both algebraically and geometrically! I will make each of the three terms represent the AREA of one of the four rectangles in this diagram. Then, I will be "*complete the rectangle".*



EXAMPLE: Given that x and y are positive integers,

solve: $\boldsymbol{x^2 + 5x^2y^2 + 20y^2 = 269}$

$x^2 + 5x^2y^2 + 20y^2 +$ ______ $= 269 +$ _____

10. *p* and *q* are non-zero integers. How many ordered pairs *(p, q)* satisfy $2pq + 2p + 3q = 18$ ?

Note: ***SFFT** also works when some terms are negative*.

11. Twice the area of a non-square rectangle equals triple its perimeter. If the dimensions are positive integers, what is the area of the rectangle?

12. Compute all integer value of $n$ between 90 and 100 inclusive that cannot be written in the form $n = a + b + ab$, where $a$ and $b$ are positive integers.

13. A, M, and C are positive integers such that A > M > C and A + M + C = 12.

If AMC + AM + AC + CM = 71, what is the maximum possible value of A?

14. If $x^5 + 5x^4 + 10x^3 + 10x^2 - 5x = 9$ and $x \neq -1$, compute the numerical value of $(x + 1)^4$.

15. Find the number of ordered pairs of integers (m, n) for which $mn \geq 0$ and $m^3 + n^3 + 99mn = 33^3$

**ANSWERS TO FANTASTIC FACTORING**

*If need assistance in the* <u>*solutions*</u> *of any of these problems, please email me at rickarmstrongpi@gmail.com or ask friends or a math teacher. Rick Armstrong*

1. $\mathbf{(4x^2 + 9y^2)(2x - 3y)(2x + 3y)}$ 2. $\mathbf{(5x + 4y)(25x^2 - 20xy + 16y^2)}$

3. $\mathbf{(2x - 3)(4x^2 + 6x + 9)}$ 4. $\mathbf{(x + y)^4}$

5. $\mathbf{(x - y)^5}$

6A. Diff. of Squares: $(x^3 - y^3)(x^3 + y^3) = \mathbf{(x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)}$

6B. Diff. of Cubes: $\mathbf{(x^2 - y^2)(x^4 + x^2 y^2 + y^4)}$

6. Proof: It is very difficult to produce the factorization of $(x^4 + x^2 y^2 + y^4)$ from 6B into the two quadratic factors of 6A: $(x^2 + xy + y^2)(x^2 - xy + y^2)$. But you can check it by expanding: $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

7A. $x^7 + y^7 = \mathbf{(x + y)(x^6 - x^5 y + x^4 y^2 - x^3 y^3 + x^2 y^4 - x^5 y + y^6)}$

7B. $x^{10} - y^{10} = \mathbf{(x - y)(x^9 + x^8 y + x^7 y^2 + x^6 y^3 + \ldots\ldots + x y^8 + y^9)}$

7C. Using the given theorem, since $y = -x$ does not make $x^{10} + y^{10}$ equal to zero, $(x + y)$ is not a factor of $x^{10} + y^{10}$ so we cannot use the given patterns to factor $x^{10} + y^{10}$ .

8 A. $4ab - 8b^2 + 3a^3 - 6a^2b = 4b(a - 2b) + 3a^2(a - 2b) = \mathbf{(a - 2b)(4b + 3a^2)}$

8B. $\mathbf{(x + 1)(y + 1)}$ 8C. $\mathbf{xyz + xy + xz + yz + x + y + z + 1}$

9. 198 ARML, Individual #1

$x(x + 1)(x + 2)(x + 3) + 1 = 379^2$ OR $x(x + 1)(x + 2)(x + 3) = 379^2 - 1^2 = 378 * 380$.

$x(x + 1)(x + 2)(x + 3)$ requires that the 4 factors be **<u>consecutive integers</u>**.

With that clue, factor: 380 * 378 = 19 * 20 * 18 * 21 and **<u>x = 18</u>**

<u>EXAMPLE</u>: Given that x and y are positive integers,

solve: $\mathbf{x^2 + 5x^2y^2 + 20y^2 = 269}$

$x^2 + 5x^2y^2 + 20y^2 + \mathbf{4} = 269 + \mathbf{4}$

$(x^2 + 4)(5y^2 + 1) = 273 = 3 * 7 * 13$

| | $x^2$ | 4 |
|---|---|---|
| 1 | $5y^2$ | **4** |
| $5y^2$ | $5x^2 y^2$ | $20y^2$ |

*By inspection, the only solution with positive integers requires* $x^2 + 4 = 13$ *while* $5y^2 + 1 = 21$ *with* ***<u>x = 3 and y = 2</u>***

10. **<u>SIX</u>**: (2, 2); ( – , 20); ( – 12, – 2); ( –5, –4); ( –3, ––8); ( –2, –21 )

11. **<u>48</u>** 12. **<u>96</u>** and **<u>100</u>** [1990 ARML, Team #7] 13. **<u>13</u>** [adapted from 2000 AMC, #12]

14. **<u>10</u>** [1994 ARML, Team #1 ]

15. **<u>35</u>** [1999 AHSME, #30] HINT: Set $s = m + n$ so that $s^3 = (m + n)^3 = m^3 + n^3 + 3mn(m + n)$And subtract the given equation from this equation. After factoring, replace s with m + n. *Good Luck!*

## TEAM ROUND – 20 MINUTES

1. The number $(9^6 + 1)$ is the product of three primes. Compute the largest of these 3 primes.

2. Of the integers between 1 and 2310, how many are divisible by exactly three of the five primes 2, 3, 5, 7, and 11.

3. If $x$ and $y$ are positive integers such that $x^2 = y^2 + 61$, find $x(x + 2) + y(y + 3)$.

4. The graph of $xy + 3x + 2y = 0$ can be produced by translating the graph of $y = k/x$ to the left and down for some constant value $k$. Find $k$.

5. Let $f(x) = x^2 + bx + 9$ and $g(x) = x^2 + dx + e$. If $f(x)$ has zeroes $r$ and $s$, and $g(x)$ has zeroes $-r$ and $-s$, compute the two roots of $f(x) + g(x) = 0$.

6. How many ordered pairs of integers (x, y) with $1 \le x \le 100$ and $1 \le y \le 100$ make the quantity $xy - x - y$ a multiple of 5?

7. If three of the roots of $x^4 + ax^2 + bx + c = 0$ are 1, 2, and 3, find the value of $a + c$.

8. x and y are real numbers that satisfy equations $x - y = 1$ and $x^5 - y^5 = 2016$. Calculate $\frac{x^5+y^5}{x+y} - (x^4 + y^4)$

9. How many ordered pairs of positive integers $(a, b)$ are that such that $\frac{1}{a} - \frac{1}{b} = \frac{1}{143}$ ?

10. Suppose that $a, b, c, d$ are real numbers such that: $ab + 3a + 3b = 216$; $bc + 3b + 3c = 96$; and $cd + 3c + 3d = 40$.

    Determine the maximum possible value of $ad + 3a + 3d$.

## ANSWERS TO TEAM ROUND

*If need assistance in the solutions of any of these problems, please email me at rickarmstrongpi@gmail.com or ask friends or a math teacher. Rick Armstrong*

1. 6481 [1992 NYSML, Team #5]
2. 186 [mathleague.org 11207, Large Team #4]
3. 2013 [mathleague.org 11607, Team #2]
4. 6 [mathleague.org 11301, Sprint #10]
5. $\pm 3i$ [1991 NYSML, Individual #2]
6. 1600 [mathleague.org 11202, Large Team #7]
7. – 61 [AHSME 1966, #30]
8. – 403 [mathleague.org 11607, Target #6]
9. 4 [mathleague.org 11308, Sprint #28]
10. 96 [mathleague.org 11307, Sprint #11]