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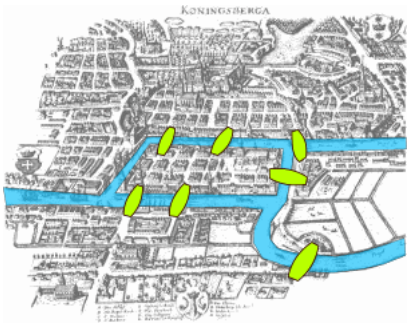
Math Circles
3/4/07

Graph Theory

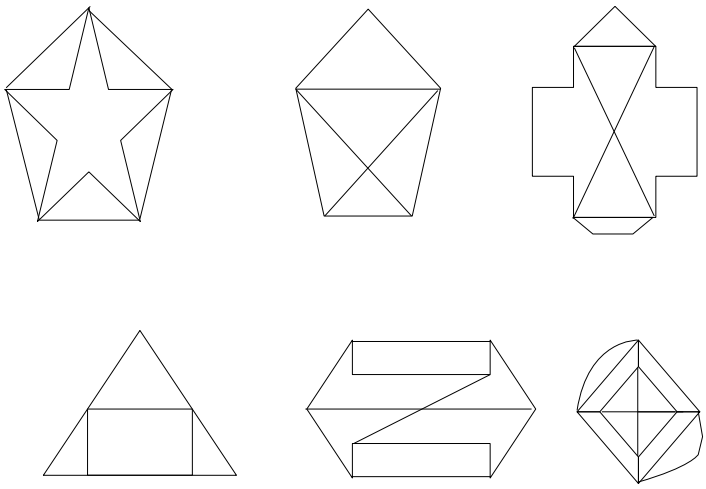
1. The Bridge Problem

The following is a diagram of the city of Königsberg and its seven bridges. Is it possible to start in one place, cross each bridge exactly once, and come back to where you started?



2. Euler Circuits

Try to find Euler circuits in the following graphs. Which graphs contain them? What sorts of patterns do you notice?



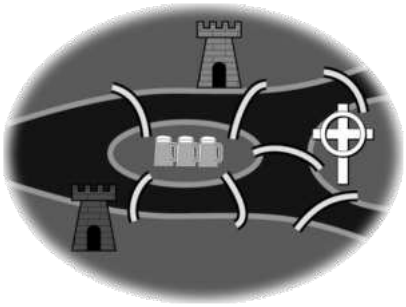
3. A Modified Bridge Problem

We would like to add bridges to the city of Königsberg to satisfy certain individuals:

- Two brothers Georg and Franz are locked in a bitter feud. First, Georg, who lives in the top castle, decides to build a bridge so that he can go for an evening stroll by crossing all the bridges exactly once, ending at the disco on the center island to go dancing. However, he wishes to build the bridge in such a way so that his brother cannot do the same. Where should he build the eighth bridge?

- As soon as Franz sees what Georg has done, he decides to build a ninth bridge so that he can do the same and tease his brother as he dances. Where should this bridge be built?

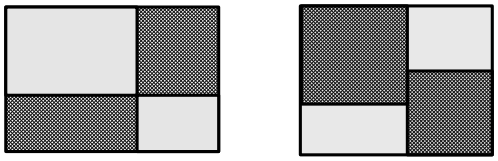
- The bishop, upon learning of this tomfoolery, shakes his head and sighs in disbelief. He then builds a tenth bridge so that everyone can take an evening stroll by crossing all of the bridges exactly once and then go dancing. Where should this last bridge be built?



This problem has to do with Euler walks, which are like Euler circuits except that it is permissible to start and stop in different places. Can you deduce a general rule for when an Euler walk is possible?

4. Coloring Maps

A proper coloring of a map is when any two countries that share a border are colored differently. A border is an entire segment, not just a corner. Of the following, the first one is a proper coloring and the second one is not.



Now consider the following question:

What is the fewest number of colors necessary to color any map?

While this question may seem straightforward, the proof of the answer is quite complicated. Many mathematicians do not accept the proof because it relies on computer computations that are too difficult to be done by hand. Still, the answer is not too difficult to guess. Try the following:

- Color the maps provided to guess an answer.
- Imagine what the worst case scenario is. That is, what kind of maps require the most colors?
- Translate this question into a graph theory question and state the problem as a mathematician would.

5. References

The two bridge diagrams and the modified bridge problem can be found in the Wikipedia article titled "Seven Bridges of Königsberg."