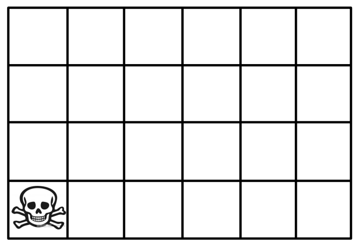
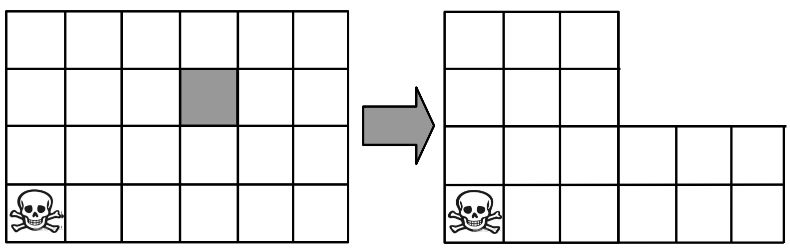
The Game of Chomp

Math Circles 2/1/09

Chomp is a two-player game. The game begins with a rectangular bar of chocolate, and the bottom left square is poisonous:



Players take turns removing pieces of chocolate. A legal move is made by choosing a square and removing everything above it or to the right of it. If the gray square is chosen below, the corresponding legal move is shown on the right:



A player loses when he or she takes the poisonous square.

Let's Play

Take a few minutes to familarize yourself with the game. Draw a few boards of various sizes and play the game with a partner.

Definition A player has a *winning strategy* if he can win the game regardless of the moves his opponent makes.

Very Thin Chomp

First we will take a look at the case where the board consists of a single row. Play games where the board has different lengths. Does either player have a winning strategy? What is that strategy?

Square Chomp

Now let's look at the case where the board is an $n \times n$ square.

1. In a game of 2×2 Chomp, does the first player have a winning strategy? Does the second player have a winning strategy? If so, describe the strategy.

2. What about in a game of 3×3 Chomp? 4×4 Chomp?

3. Make a conjecture about which player, if any, has a winning strategy for $n \times n$ Chomp. Can you describe the winning strategy?

Thin Chomp

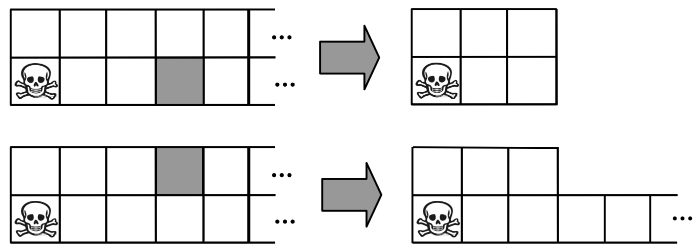
Now we will switch gears from the square case to the $2 \times n$ case.

1. Does either player have a winning strategy in 2×3 Chomp? If so, describe the strategy.

2. What about in a game of 2×4 Chomp? 2×5 Chomp?

3. Make a conjecture about which player, if any, has a winning strategy in $2 \times n$ Chomp. Can you describe the winning strategy?

4. Instead of looking at a finite board, we could look at a board of infinite length. Moves are made in the same way. The $2 \times \infty$ board is shown below, along with two possible opening moves.



Is it possible to modify the $2 \times n$ strategy for the $2 \times \infty$ board? If so, what modifications must be made?

Bonus Question: In a game where the board is of arbritrary size, which player ought to have a winning strategy? Can you prove that is the case?
(Hint: suppose that the other person has a winning strategy. Can you utilize that strategy in some way so that you win the game instead?)

Thicker Chomp

We would like to be able to find winning strategies for boards that are arbitrarily large. In 2002, a winning strategy for the $3 \times n$ case was found by a high school senior, Steven Byrnes, as part of a larger theorem, but it is difficult to find. An easier question is this: what sorts of moves could the first player make so that the second player would then have a winning strategy?

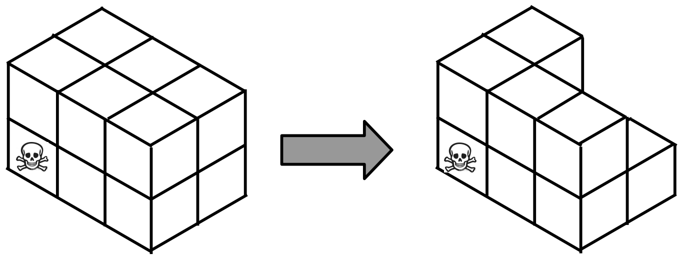
How does knowing that the first player has a winning strategy for any rectangular board affect the way you play?

Are there board positions from which you know you will lose with perfect play?

If you are interested, try to find a winning strategy for a $3 \times n$ or even a $4 \times n$ board. Try starting with small boards.

3D Chomp

In 3D Chomp, the chocolate bar becomes a chocolate block. The poisoned cube is the front left bottom cube. This time, a legal move is made by selecting a cube and removing everything above, behind and to the right of it. An example is shown below:



As before, the player who removes the poisoned cube loses.

1. Build a $1 \times 2 \times 2$ board. Which player will win this game? Why? In general, which player will win a $1 \times m \times n$ game of 3D Chomp?

2. Which player will win 3D Chomp in general? Can you prove it?

3. Build a $2 \times 2 \times 2$ board. Can you find a winning strategy?

4. What about a $3 \times 3 \times 3$ board?

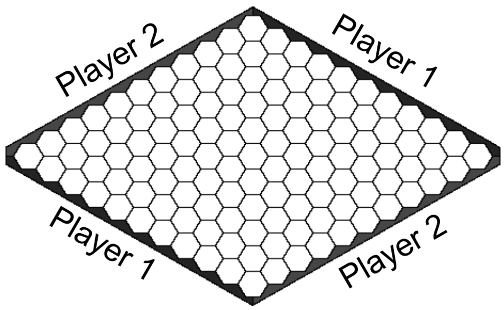
5. Can you find a winning strategy for an arbitrary cube?

6. Can you find winning strategies for other board shapes?

Other Games

Hex

The game of Hex is also a two player game. Players take turns placing stones, and the first player to connect his or her assigned sides of the board wins.



Playing on an $m \times n$ board means that the first player's side is made up of m hexagons and the second player's sides is made up of n hexagons.

1. Play several games on small boards where $m > n$. Which player wins? Why is that the case? Can you find a winning strategy for that player?

2. Now play games where $m < n$. What happens in this instance?

3. The case where $m = n$ is generally more difficult. The case where $m = n = 2$ isn't difficult, though. Who wins in this instance?

4. Can you find a winning strategy for the 3×3 case?

5. Make a conjecture about which player has a winning strategy. Can you prove it using an argument similar to that in Chomp?

6. Do the same for larger boards. As the size of the board increases, it becomes much more difficult to find a winning strategy. Currently, strategies are only known up through the 9×9 case.

Colonel Blotto

Colonel Blotto is a two player game in which each player has m divisions and the two players are going to wage battles on n different battlefields. Players divide their divisions between the different battlefiels and then wage war. A player wins the battle if he or she has more divisions there. 2 points are awarded to a player for winning a battle, 1 for a tie, and 0 for losing the battle. The player with the most points wins.

For example, if there are 100 divisions to spread over 9 battlefields, one possible game could look like this:

11	11	11
11	11	11
11	11	12

Player 1

20	0	20
0	20	0
20	0	20

Player 2

In that game, Player 1 earned 8 points and Player 2 earned 10, so Player 2 wins.

1. Is there a winning strategy in this game? Why or why not?

2. Are there ways of positioning your troops so that you are guarenteed to lose?

3. Are there ways of positioning your troops so that you will likely lose?

4. Suppose you have four divisions and two battlefields. Is there a way of positioning your troops so that you would win?

5. Alter the number of divisions and battlefields, and try to find positions with the greatest chance of winning and the greatest chance of not losing.