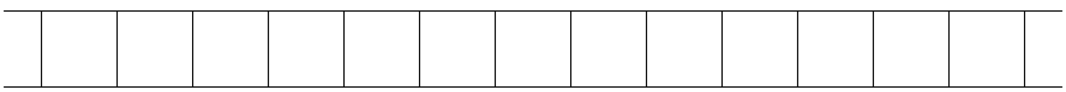
Conway's Game of Life

Geir Arne Hjelle

Math Circle, February 22nd, 2009

1 One dimensional life

We will first look at life on a long strip of cells, which we will call our *universe* (the technical name is cellular automaton). We will draw the cells as squares like this



Each cell has one of two states: It is either *alive* (marked as a filled in or crossed out cell), or *dead* (marked as an empty cell). In the following example the second, fourth, fifth, seventh, eight, ninth, twelveth and thirteenth cells are alive, while the others are dead.

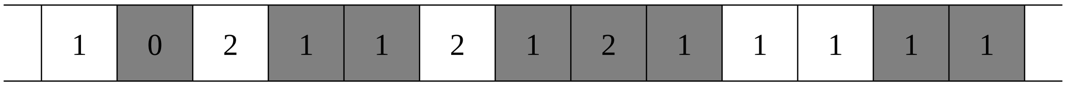
We will now bring our cells to life! This is done by specifying simple rules about how each cell can die or come to life. Our set of rules are (a neighbour is simply an adjacent cell)

- An alive cell dies of isolation if it has no alive neighbours.
- An alive cell stays alive if it has exactly one alive neighbour.
- An alive cell dies of overcrowding if it has two alive neighbours.

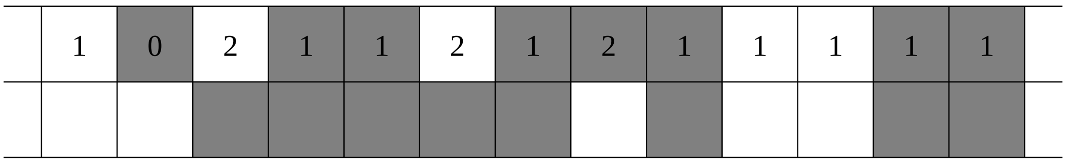
and

- A dead cell stays dead if it has zero or one alive neighbour.
- A dead cell comes alive if it has two alive neighbours.

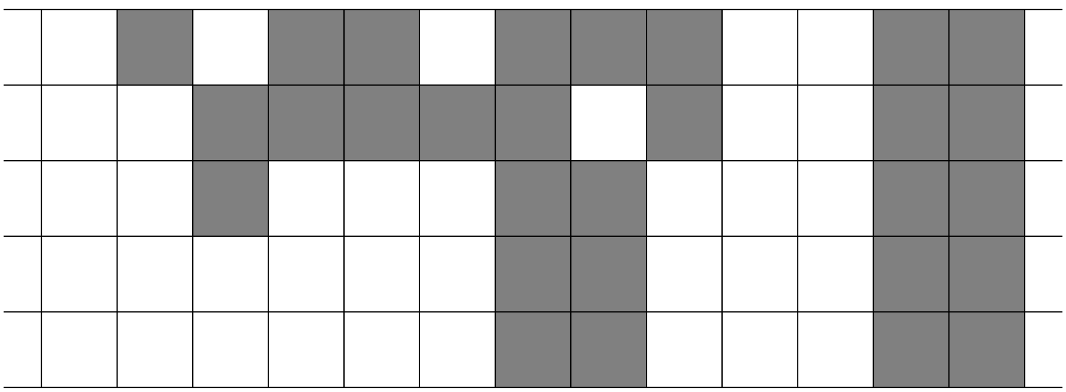
To find the second generation for the cells above, we first count how many alive neighbours each cell has.



Next, we draw the second generation below the first one according to the rules above.

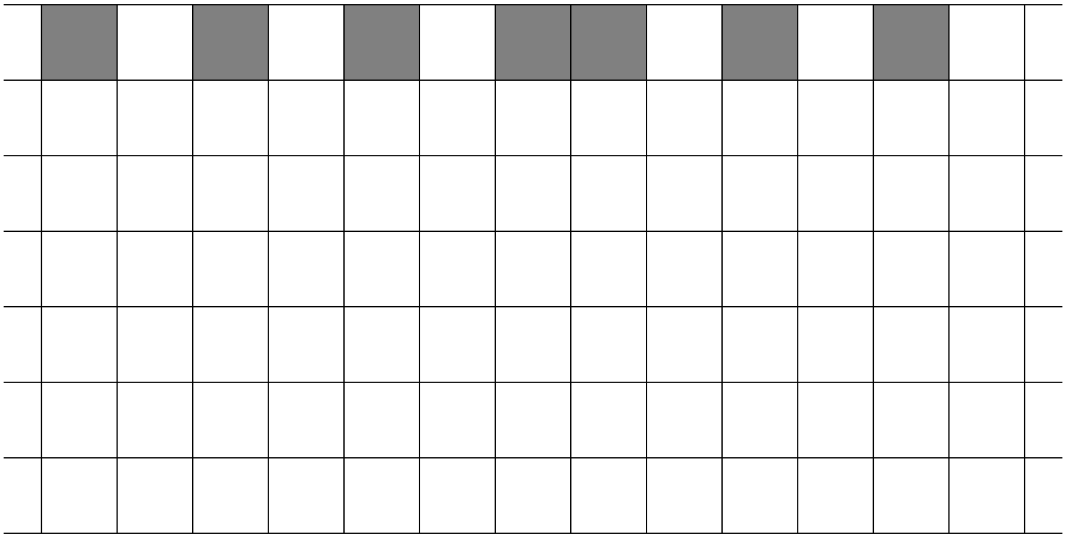


Once we have found the second generation we find the third generation the same way, and so on. Below we have drawn the first five generations.

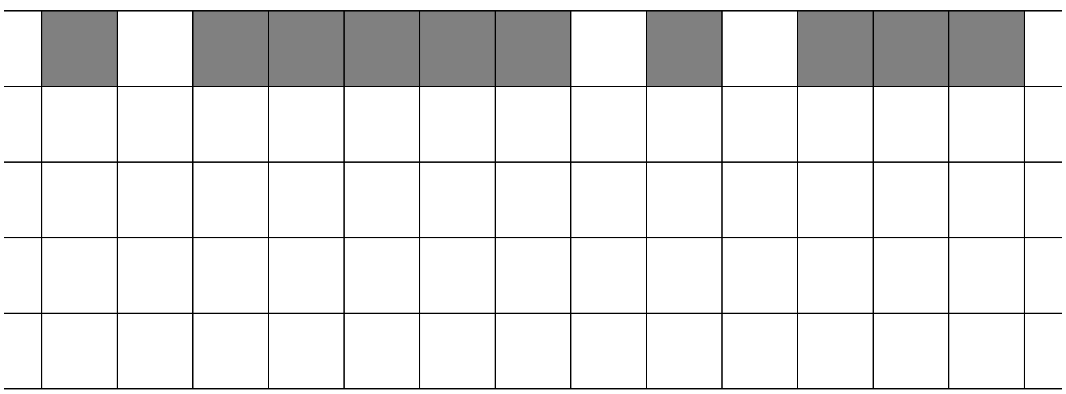


Note that the fourth and the fifth generations are the same. Since we will keep applying the same rules over an over again, all subsequent generations will also be the same. We say that the system has reached a *stable* state.

Question.
Find the next generations of the cells below.



Question.
Find the next generations of the cells below.



After trying a few examples like these we are starting to get a feeling for how different cell patterns evolve, and we can start to ask more complicated questions.

Question.
Is the □□■■□□-pattern the only stable pattern, or can you find others?

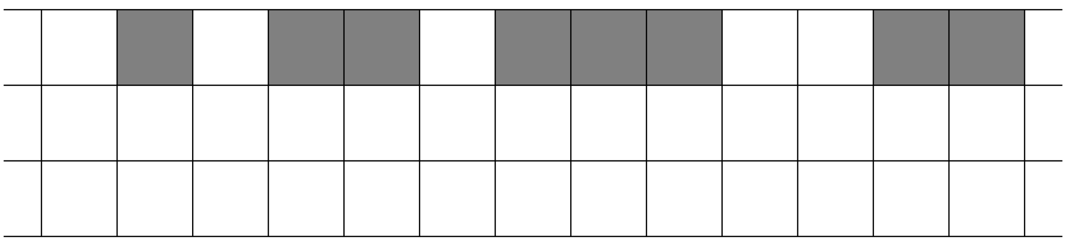
Question.
Can the universe expand? (Remember that the universe is the strip of cells. You should read this question as "Can new cells come alive outside our original set of alive cells?")

2 Changing the rules

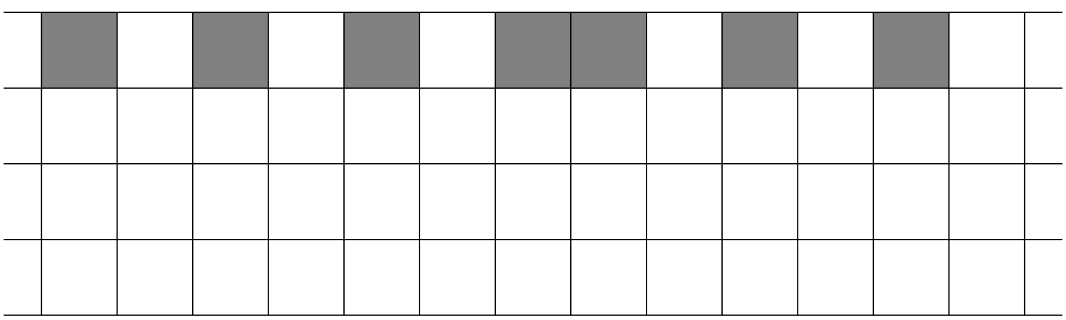
We will first agree on a compact notation for the rules for how the cells die and come alive. Note that everything is decided in terms of how many neighbours a cell has, and that each cell has only two options, it is either alive or it is dead. We can therefore specify the rules simply by specifying for how many alive neighbours an alive cell survives, and for how many alive neighbours a dead cell comes alive. For instance, the rules we have used so far can be written 1/2.

For a first example of different rules, let us consider 12/2. This should be interpreted as a living cell with 1 or 2 living neighbours stays alive, and a dead cell with 2 living neighbours comes alive.

Question.
Find the next generations of the cells below, using the 12/2-rules.



Question.
Find the next generations of the cells below, using the 12/2-rules.



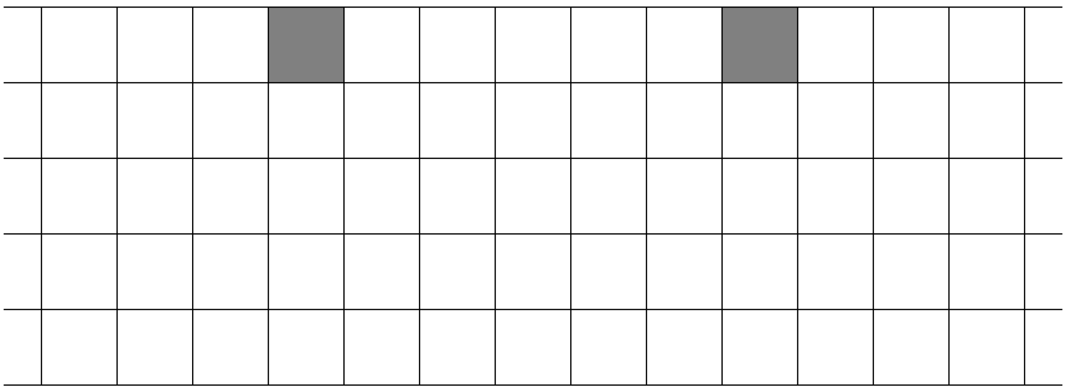
Question.
What are the stable patterns in the universe using the 12/2-rules?

Question.
How would you say the behaviour of the cells changed when we changed the rules?

3 Exploding the universe

Instead of changing the rule for survival like we did above ($1 \to 12$), we can change the rule for how new cells come alive. Let us now use the rules 1/12. That is, a living cell with 1 living neighbour stays alive, while a dead cell with 1 or 2 living neighbours comes alive.

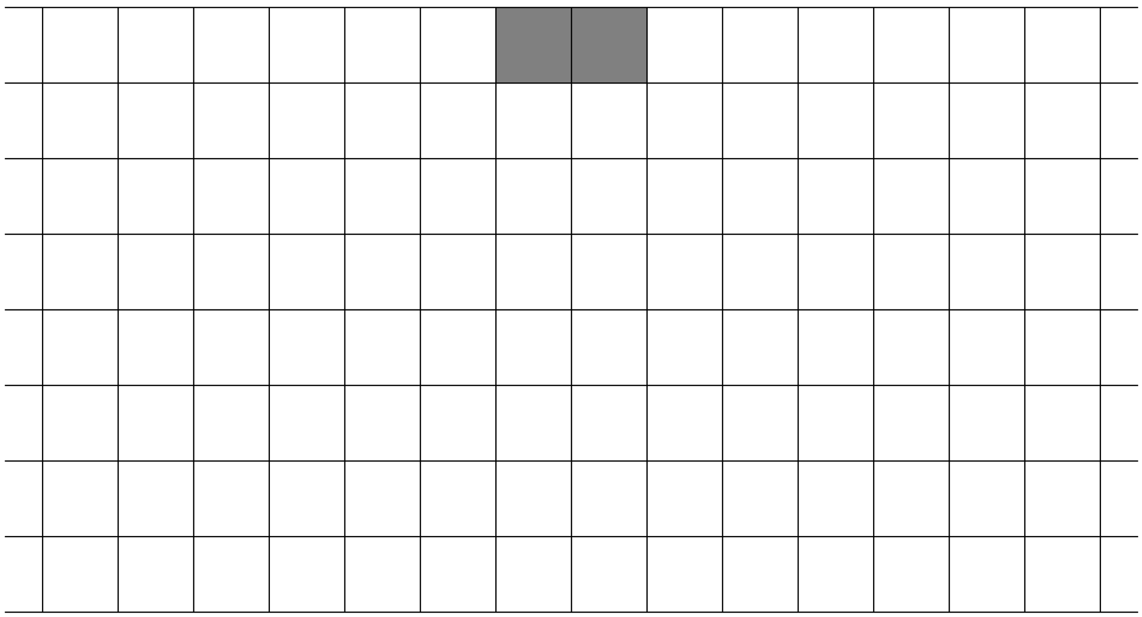
Question.
Find the next generations of the cells below, using the 1/12-rules.



The pattern ⋯□■□■□■□⋯ will change to ⋯■□■□■□■⋯, and then back again. We call such patterns *stable* with a *period* of 2.

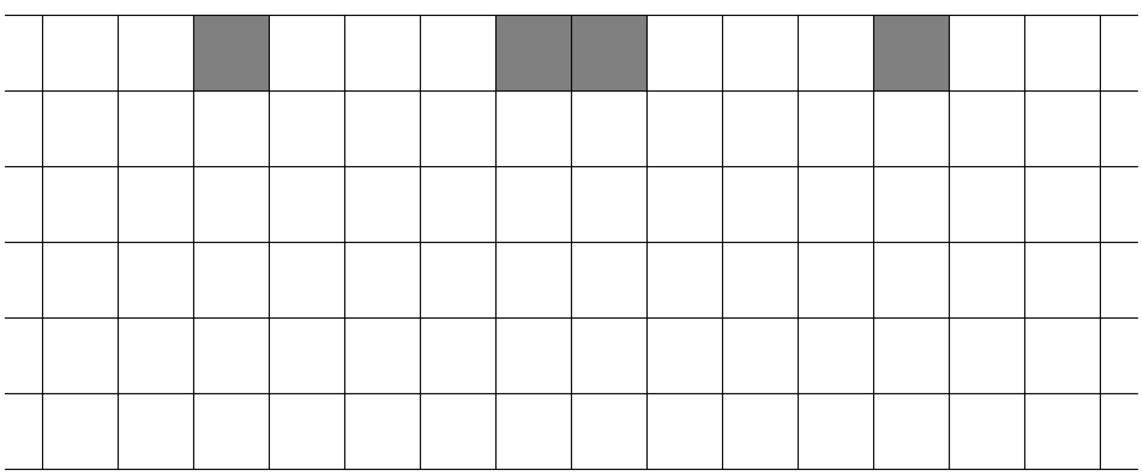
Question.
Can the cell population die out in the 1/12-universe?

Question.
You may have seen the following pattern in earlier math circles? Again, use the 1/12-rules.



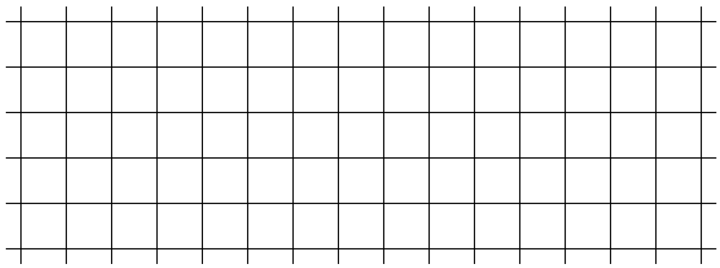
Question.
What would you say is the characteristic behaviour of the cells under the 1/12-rules?

Question.
For another example, let us use the /1-rules. That is, all living cells die, and dead cells with 1 alive neighbour comes alive.

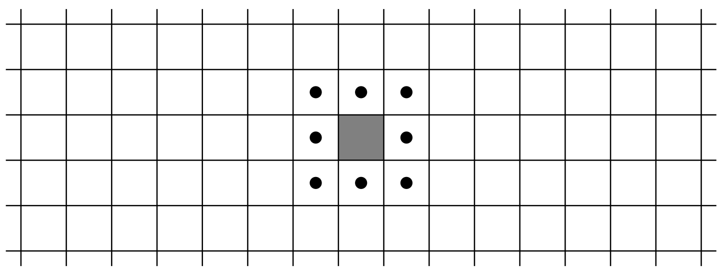


4 Breaking into the second dimension

We will now begin using two-dimensional universes. Instead of living on a strip, our cells will now live on a grid of squares.



Each cell now has eight neighbours.



We will continue to state the rules in the same manner as before. Consider the rules 23/3:

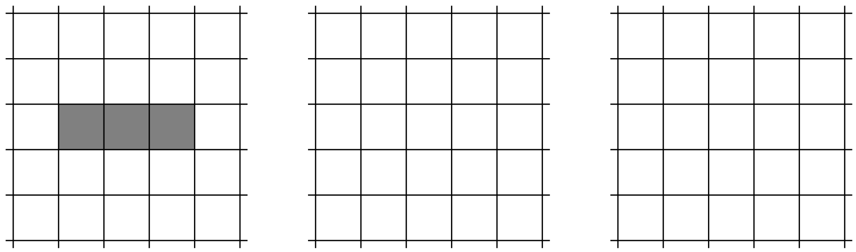
- An alive cell dies of isolation if it has zero or one alive neighbour.
- An alive cell stays alive if it has two or three alive neighbours.
- An alive cell dies of overcrowding if it has four or more alive neighbours.

and

- A dead cell stays dead if it has two or less, or four or more alive neighbours.
- A dead cell comes alive if it has exactly three alive neighbours.

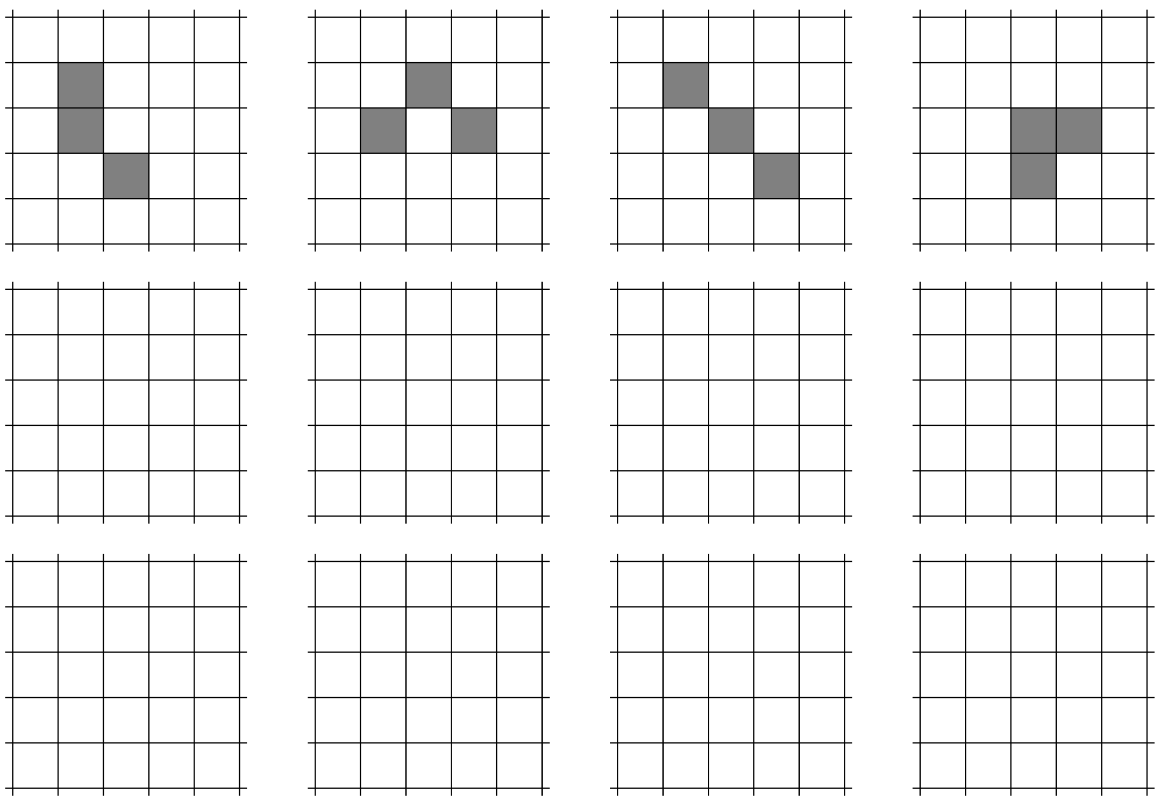
We need at least three alive cells before anything interesting happens (because an alive cell needs two neighbours to survive, and a dead cell needs three neighbours to come alive.

Question.
Find the first few generations below.



The pattern above is the simplest example of what is called a *blinker*. It is stable with period 2.

Question.
What is the fate of the other 3 cell populations?

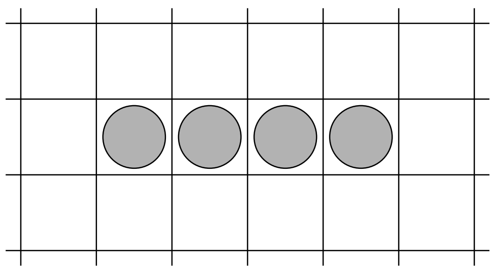


5 The Game of Life

The two-dimensional universe with the 23/3-rules is called The Game of Life, because it exhibits amazingly complex behaviour, where different cell patterns combine and interact in often mysterious ways. It was discovered by John H. Conway, and popularized by Martin Gardner in 1970. In the almost 40 years that have passed since then, the Game of Life has been intensely studied and many interesting and fascinating cell patterns have been found, many of them with the help of computers.

The easiest way to manually (without computers) explore the Game of Life is to use a large checkerboard (like a chess- or go-board), and small flat counters of two different colors. We will the Universe of Life-board, and pennies and dimes.

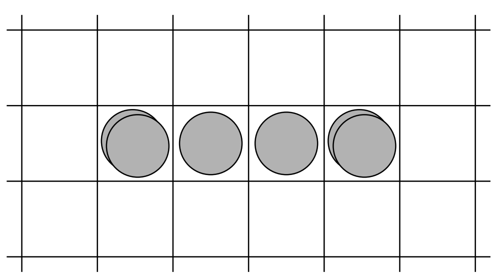
On the grid, lay out four pennies in the following pattern.



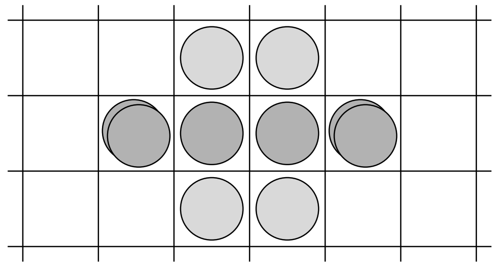
Now we want to move the pennies to the next generation. The problem is that all the cells are alive or dead in one generation based on the cells in the previous generation. Once we start moving the pennies around, we are bound to forget the states of some of the cells in the previous generation.

Instead we use the following procedure.

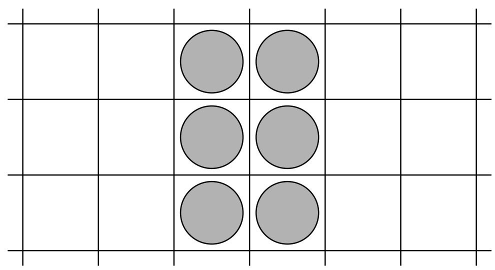
- Identify the living cells that will die, and place second pennies on top of the ones that are already there.



- Identify the dead cells that will come alive, and place one dime in each of those cells.

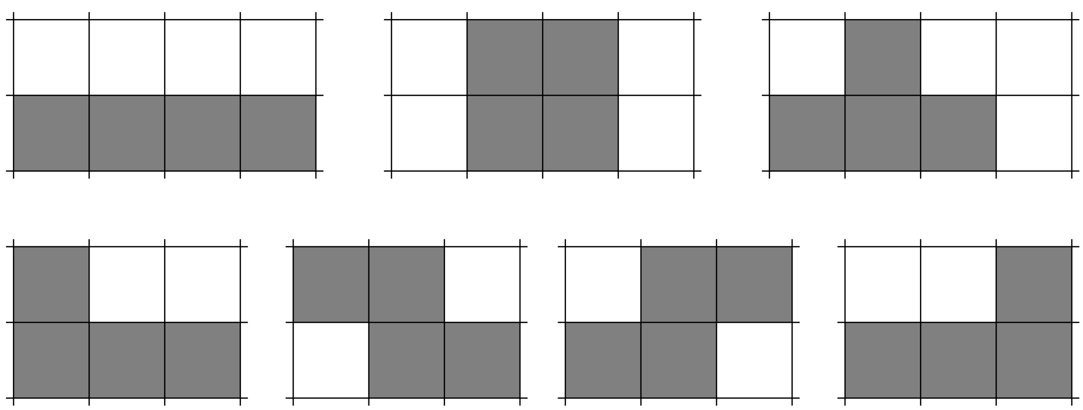


- Remove the dead pennies, that is the cells containing two pennies, and exchange the newborn dimes into pennies.

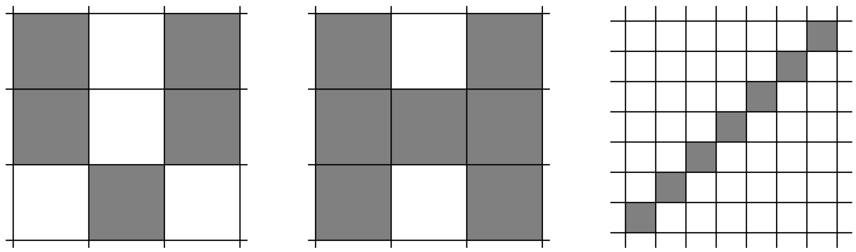


This gives us the next generation, and minimizes the risk that we make an error.

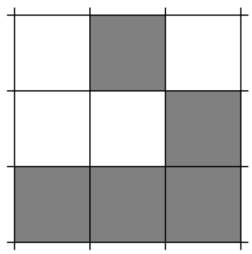
Question.
What is the fate of the seven tetrises?



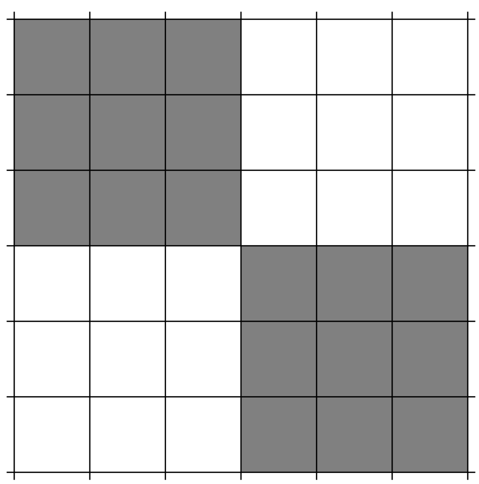
Question.
What happens with the V, the H, and the /?



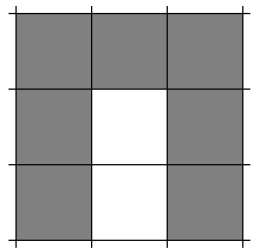
Question.
The following pattern is called a glider. Can you figure out why?



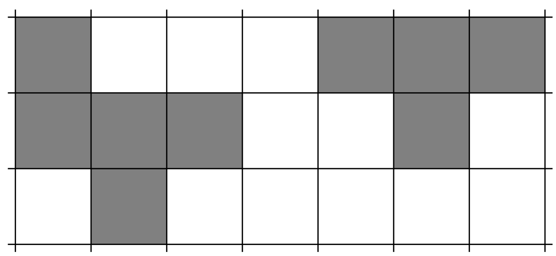
Question.
The Figure 8 is stable with period 8!



Question.
The pi (π) is, as the number, a fascinating creature. What happens with it?



Question.
The rabbits is another cell pattern that displays a fascinating behaviour. It will not settle into a predictable state for more than 17000 generations!



6 Computer simulation

The computer is an indispensable tool when exploring the Game of Life. Once you have gotten to know the basic patterns the computer can be used to figure out how the more advanced patterns evolve (and the complex patterns can start out surprisingly innocent looking, the pi does not became stable until its 173rd generation!).

There is much material about the Game of Life online, including programs that can simulate the game. One nice online Game of Life simulator is

`http://www.bitstorm.org/gameoflife/`

An extensive catalog over interesting Game of Life patterns can be found at

`http://radicaleye.com/lifepage/patterns/contents.html`

Finally, programs that can be installed on your computer (which will be faster for huge simulations) can be found online. One of them is Golly (which includes an amazing hyper-speed mode).

`http://golly.sourceforge.net/`

Universe of Life