# How to Count

## 1. Permutations and Combinations

**Definition 1.** A *permutation* of a collection of objects is a list which contains each of the objects exactly once in some order. For example, 13245 and 54321 are both permutations of the numbers 1, 2, ... 5.

**Exercise 1.** *Write down all the permutations of the letters a, b, c.*

**Exercise 2.** *How many distinct permutations are there of the the numbers 1 through 4? There are more than 100 permutations of the numbers 1 through 5, can you determine exactly how many there are without writing them all down?*

At this point we should introduce some important notation:

**Definition 2.** For $n$ a natural number, $n!$ (pronounced "n factorial") is the product of the first $n$ natural numbers, i.e., $n! = n * (n-1) * (n-2) * \cdots * 2 * 1$. For convenience we also define $0! = 1$.

For example, $5! = 5 * 4 * 3 * 2 * 1 = 120$.

**Exercise 3.** *Suppose we want to place rooks on a 3 x 3 chessboard so that each row and column of the board contains exactly one rook. One such position is shown below.*

| | | |
|---|---|---|
| | $R$ | |
| $R$ | | |
| | | $R$ |

*There are 5 more. Write them down. Can you do this in an organized fashion? How do you know there aren't any more?*

**Exercise 4.** *How many ways are there to place rooks on a 4 x 4 board so that each row and column contains exactly one rook? On a 5 x 5 board? An n x n board?*

**Exercise 5.** *Suppose there are 20 students in class. How many ways are there to choose a class president and vice president, assuming that one student can't hold both offices?*

**Exercise 6.** *Now suppose another class of 20 students has a more egalitarian mindset and so instead of having a president and vice president they have two 'co-presidents' of equal rank. How many ways are there to choose the two co-presidents? How is this different from the previous exercise?*

**Exercise 7.** *How many ways are there to choose a president, vice president, and treasurer from a class of 20 students, again assuming no student can hold more than one position?*

**Exercise 8.** *How many ways are there to choose a committee of 3 students from a class of 20? (Hint: Think about how this relates to permutations and to the previous exercise.)*

**Exercise 9.** *How many ways are there two choose a committee of 2 students out of a group of 10? A group of 100? How about a group of n students?*

**Exercise 10.** *How many ways are there to choose a committee of 4 students out of a class of 10? A committee of 5 students? (Just find and expression/explain how you'd find it. You don't need to actually multiply/divide it all out.)*

**Exercise 11.** *How many ways are there to choose a* $k$ *person committee from a class of* $n$ *students?*

## 2. Pascal's Triangle

In combinatorics the number of ways to choose $k$ elements from a set of $n$ (which you found a formula for in the last exercise) is denoted $\binom{n}{k}$, which is pronounced "n choose k." One nice way to find $\binom{n}{k}$ is called Pascal's Triangle. Start with a triangle like this:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

Now add a third row underneath as follows. First, pretend there are zeros at both ends of the second row, then make the third row by adding adjacent entries together and placing this sum below and between them:

$$\begin{array}{ccccccc} & & & 1 & & & \\ 0 & & 1 & & 1 & & 0 \\ & 1 & & 2 & & 1 & \end{array}$$

We can keep adding rows by the same process (we don't write the zeroes). The first 4 rows look like this:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

**Exercise 12.** *Write out the first 7 rows of Pascal's triangle.*

**Exercise 13.** *Calculate* $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$ $\binom{4}{3}$ *and* $\binom{4}{4}$. *Examine Pascal's triangle. Notice anything interesting?*

**Exercise 14.** *From your observation in the last exercise, see if you can find* $\binom{6}{3}$ *just by looking at the triangle. Now check that your answer is correct.*

**Exercise 15.** *Find* $\binom{5}{2}$ *and* $\binom{5}{3}$ *in Pascal's triangle. Compare to the location of* $\binom{6}{3}$. *What's the relationship between these three numbers?*

**Exercise 16.** *Remember that* $\binom{5}{2}$ *is the number of ways to choose 2 elements from a set of 5,* $\binom{5}{3}$ *is the number of ways to choose 3 elements from a set of 5, and* $\binom{6}{3}$ *is the number of ways to choose 3 elements from a set of 6. Explain why the relationship in (**??**) makes sense in terms of these definitions.*

**Exercise 17.** *Can you find* $\binom{n}{k}$ *just by looking at* $\binom{n-1}{k-1}$ *and* $\binom{n-1}{k}$*? Explain why, and write down the relationship between these three numbers. What does this say about or trick with Pascal's triangle? Will it always work?*

## 3. Inclusion-Exclusion

**Exercise 18.** *How many numbers between 1 and 100 are not divisible by 2? By 3? By 5?*

**Exercise 19.** *How many numbers between 1 and 100 are divisible by neither 2 nor 3?*

**Exercise 20.** *How many numbers between 1 and 100 are divisible by at least one of 2, 3 or 5? (It might help to consider the Venn diagram below.)*