Pick's Theorem

- Today we will study what are called "lattices" in mathematics. Have you ever been to an apple orchard? The trees are planted in a grid (represented by dots). A farmer had an apple orchard. A bunch of goats ate the stock of apples. He then fenced in those trees already eaten as represented in Figure 1. He needed to calculated the enclosed area.

  There is a very simple formula to calculate the area inside a lattice polygon once we know two pieces of information:

  1. The number of vertices on the boundary of the lattice polygon.
  2. The number of vertices that lie completely inside the polygon.

  Our goal today is to guess what the formula should be, then show why the formula holds.

- 1. On Page 2 complete the drawings for rectangles of the form $1 \times n$.
  2. Complete the chart below using the pictures you've just drawn.

Figure	Area	Boundary Vertices=B	Interior vertices=I
1×1			
1×2			
1×3			
1×4			
1×5			

  3. Suppose we take a general rectangle of the form $1 \times n$. What is the area? How many boundary vertices are there? How many interior vertices are there?
  4. How is A calculated from B and I? Make your own guess for a formula. My guess is $A = \frac{1}{2}B - 1$. Do you have another guess?
  5. Draw a 2×2 rectangle. What is A? What is B? What is I? Does my formula hold in this case? Does your formula hold?

- 1. Let's make things a little more difficult by doing calculations for rectangles of the form $2 \times n$. Complete the table below by completing the drawings on page 3.

Figure	A	B	I	$\frac{1}{2}B - 1$	$A - (\frac{1}{2}B - 1)$	Your Guess Here
2×1						
2×2						
2×3						
2×4						
2×5						
2×6						

2. Did your guess always work? If not, don't worry. Look the I column and the $A - (\frac{1}{2}B - 1)$ column. What can you say about those columns?

3. From these observations, make a new guess for the area formula as a function of B and I. Check with an instructor to make sure you have the correct guess.

- 1. Look at the 6 figures on page 4. For the polyhedra P, Q, R, S, T, and U, calculate the area as well as the expression $\frac{1}{2}B + I - 1$ to fill out the table below.

Figure	A	B	I	$\frac{1}{2}B + I - 1$
P				
Q				
R				
S				
T				
U				

2. Do you think our new guess is the correct formula?

- 1. If we are given a polyhedra P, let $Pick(P) = \frac{1}{2}B_p + I_p - 1$ where B_p is the number of boundary vertices and I_p is the number of interior vertices. We have reason to believe the following theorem, known as Pick's Theorem, is true:

Let P be a lattice polygon. Then $Area(P) = Pick(P)$.

2. On Page 5 calculate $Pick(P)$ and $Pick(Q)$.

3. Now consider the figure R obtained by gluing P and Q together along the boundary indicated by the dotted line. The dotted line is not part of the polyhedra R, but is there only to show how R is obtained from P and Q. Calculate $Pick(R)$

4. Add $Pick(P)$ and $Pick(Q)$. Compare your answer to $Pick(R)$ What do you notice?

5. In this example, we've shown $Pick(P)+Pick(Q) = Pick(R)$. Can you explain why this will always be true if we glue two polyhedra together along a common boundary?

6. Here is a nice fact: every lattice polyhedra can be subdivided into triangles. Verify this for figures P, Q, R, S, T, and U on Page 4.

7. We are "driving" towards a way to show Pick's theorem is true. We know that the Pick formula is additive and that every polyhedra can be decomposed into triangles. Using these two facts, all we have to do is show that $Pick(T) = Area(T)$ for a triangle T. Can you explain why?

8. By applying the additivity formula show that if we know $Area(T) = Pick(T)$ for a right triangle, then we have $Area(P) = Pick(P)$ for an arbitrary triangle. To do this look at Page 6. Shown are three triangles T, R, and S. Show $Pick(T) = Area(T)$, $Pick(R) = Area(R)$, and $Pick(S) = Area(S)$ assuming $Pick = Area$ for right triangles. At this point, no numbers should be involved. I'll get you started in showing that $Pick(T) = Area(T)$. On Page 6, I've redrawn the triangle T and made a bigger right triangle U with T inside. The triangle Q is also a right triangle. By our assumption on right triangles we get

$$Pick(U) = Area(U) \text{ and } Pick(Q) = Area(Q)$$

We also can use the fact that $Pick$ and $Area$ are additive to get

$$Pick(U) = Pick(T) + Pick(Q) \text{ and}$$
$$Area(U) = Area(T) + Area(Q)$$

Now use these equations to show $Area(T) = Pick(T)$. Do similar arguments for the figures R and S.

9. Now we have reduced our proof of Pick's Theorem to the case of a right triangle. Our next step is to reduce down to a rectangle. On page 7 a rectangle is shown. Suppose we know $Area(R) = Pick(R)$. Show that $Pick(T_1) = Area(T_1)$. Again, no numbers should be involved, but use reasoning like in the previous problem.

10. Now we've reduced Pick's theorem to the case of a rectangle. How do we finish the proof?

- Next consider the figure on Page 8. What is the area A? What is the $Pick$ function of this figure. What is wrong? What hidden assumption did we make in our proof of Pick's Theorem? Can you fix the formula for this case?

- Does there exist a version of Pick's Theroem for polyhedra whose vertices lie on a three-dimensional lattice?