Russian Circle
Circle Problems

Math Circle - Summer 2009

Instructions: Work as many problems as you can. Some of these problems are quite difficult. Even if you can not completely solve a problem, try to learn as much as you can about the problem.

1. How many zeros are at the end of the decimal representation for the number

$$100! = 100 \times 99 \times 98 \times 97 \times \cdots \times 2 \times 1$$

2. Find the last digit of the number 2009^{2009}.

3. Prove that for any natural number n, the number

$$n^3 + (n + 1)^3 + (n + 2)^3$$

   is divisible by 9.

4. Let p be a natural number and consider the number formed by writing down $2n$ ones:

$$p = 1111\ldots1111 \quad (2n \text{ ones})$$

   Is p prime or composite? Justify your answer.

5. You have \$127 and 7 envelopes. You are to put the money into the envelopes in such a way that you can pay any bill between \$1 and \$127 without opening the envelopes.

   (In other words, you just hand the envelopes to make the payments. And, the bills to be paid must be integers.)

6. PROFESSOR AND UMBRELLA

   A professor has an umbrella in the office and another one at home.

   If it is raining when the professor goes out, he takes an umbrella with him, if one is available. If it is not raining, he doesn't carry any umbrellas with him. Whenever the professor walks with an umbrella, he stays dry. Whenever he walks without an umbrella in the rain, he gets wet.

   Let p be the probability of rain (which is the same on every trip). What is the (steady state) probability of getting wet? I.e., what is the probability of rain and no available umbrella?

7. A bus, a truck and a sedan are driving on a perfectly straight highway, in the same direction at constant speeds (so none of these are speeding up or slowing down, but they are not necessarily driving at the same speeds).

   At some point the bus is ahead, the sedan is behind and the truck is in between. At this point, the distance between the bus and truck and the distance between the truck and sedan are the same. Ten minutes later, the sedan passes the truck. Another 5 minutes later, the sedan passes the bus.

   How long will it now take for the truck to pass the bus?