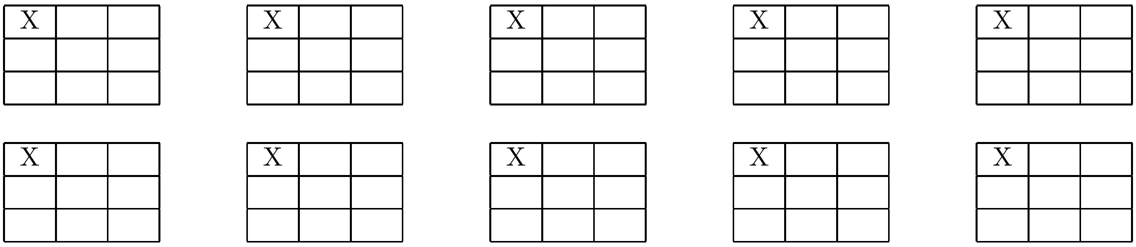
The game of chocolate is played as follows. The game board is a rectangular grid of square pieces of chocolate, one of which is marked as 'rotten'. Players take turns by making a cut along a row or column, and removing the section that does not contain the rotten piece. The goal is to leave your oponent with only the 'rotten' piece left.

Play some 2x2 games:

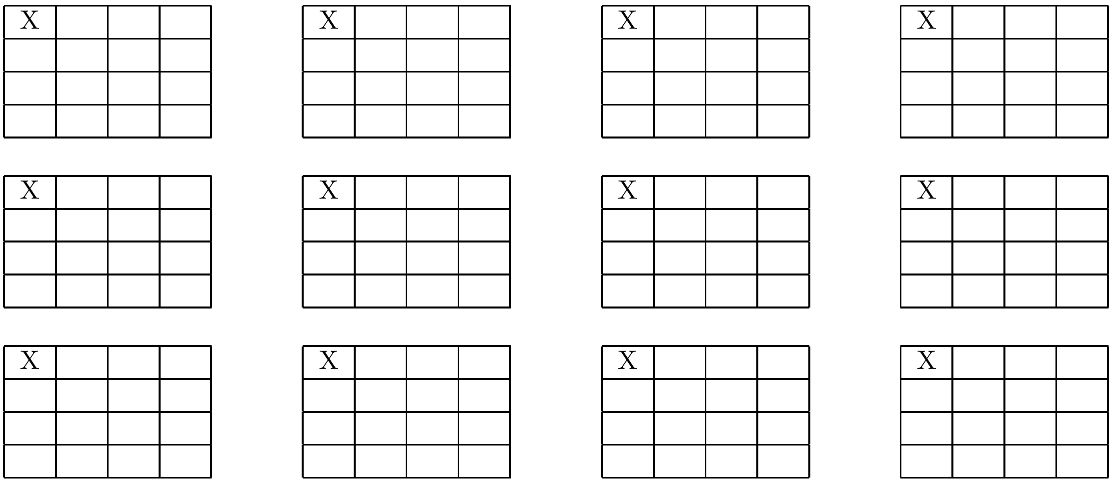
Can you determine whether the player going first or second should win?

Now let's play 3x3:



If you're the second player, can you find a winning strategy?

What about 4x4?



With 'perfect play', which player should always win? What about in the 10x10 case?

Let's change things up by moving the 'rotten' piece around.
Try the 3x3 case again:

	X	

	X	

	X	

	X	

	X	

Now 5x5:

		X		

		X		

		X		

		X		

		X		

		X		

		X		

		X		

		X		

		X		

		X		

		X		

Which player should always win, the first or the second player?

Try 7x7 if you need:

			X			

			X			

			X			

			X			

			X			

			X			

Which player should always win, the first or the second player? What is their strategy?

Now let's move the rotten piece around some more.

Try this 5x5 a few times:

	X			

	X			

	X			

	X			

	X			

	X			

	X			

	X			

And this 7x7:

	X					

	X					

	X					

A different 7x7:

		X				

		X				

		X				

Let's try a 4x4 again:

	X		

	X		

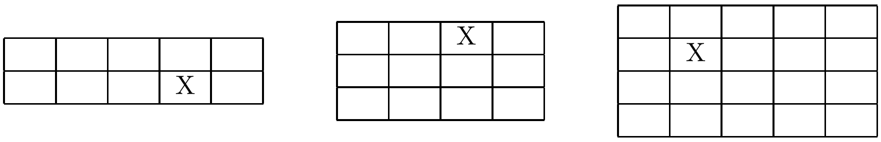
	X		

	X		

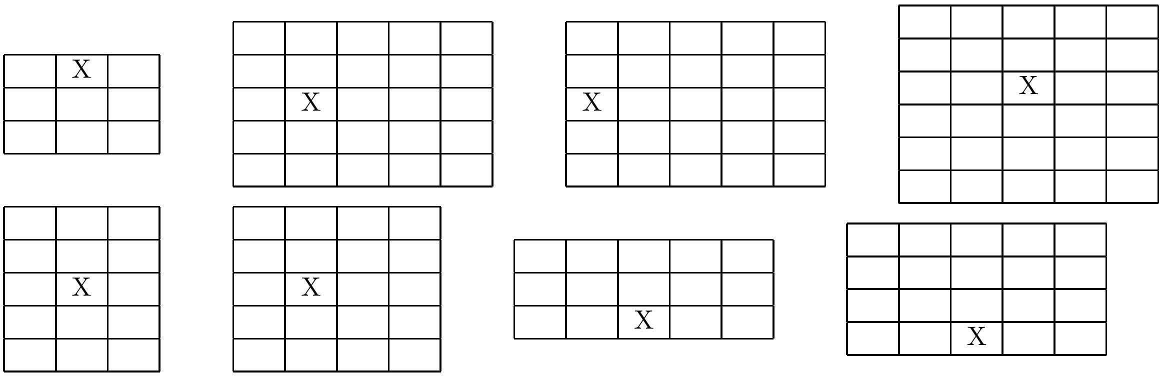
If the rotten piece lies on the diagonal, which player can win with flawless play?

Can we write down a rule to summarize all of these 'losing positions' for the first player? What is the second player's 'winning strategy'? (**Hint:** Look at the distance from the X to the edge of the board in each direction)

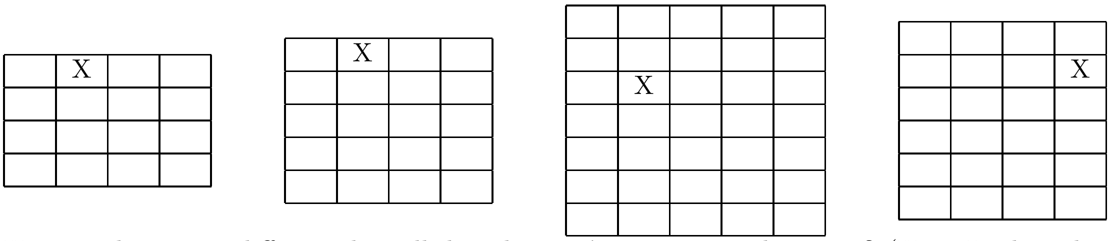
Now that we've established lots of losing positions for the first player, let's look at some winning ones. In each of the following, what move should the first player make?



Identify which player should win in each of the following:



What's the common strategy for all of these cases? Will your strategy work in any of the following cases?



How are these cases different than all the others we've seen up to this point? (Hint: Look at the distances from the X to the edges)

It turns out this is the hardest case (and the only case left). We'll come back to this later.

Let's play a new game, called Nim. In this game, we have 4 piles of stones. Each player takes turns removing as many stones from a single pile as they want (but only from one pile). The winning player is the one who takes the last stone.We'll use the notation (3)(4)(4)(8) to denote the game with one pile of 3 stones, two piles of 4 stones, and one pile of 8 stones.

Play the following games of Nim several times:

1. (0)(1)(2)(2)

2. (1)(2)(2)(3)

3. (0)(2)(2)(4)

4. (2)(2)(2)(3)

5. (1)(1)(2)(2)

6. (1)(2)(2)(2)

7. (0)(2)(2)(2)

8. (0)(2)(2)(3)

Which player should win in each of these games? Can we see a winning strategy?

Alternate playing Nim and Chocolate with the following games

(1)(1)(1)(1)

	X	

(1)(1)(2)(3)

	X		

(0)(0)(2)(2)

		X

(1)(2)(2)(2)

		X	

(1)(2)(2)(3)

	X		

(0)(1)(2)(3)

	X		

Do we notice any similarities between the two games? How do our winning strategies for the two games compare?

At this point, we've seen that our games of Nim and Chocolate are equivalent. We agreed earlier that there was only one hard case of Chocolate. Let's see if we can make sense of that case in Nim.

Try these Nim games, looking for a best strategy:

1. (0)(1)(2)(3)
2. (1)(3)(4)(9)
3. (2)(10)(15)(16)

If we were exceptionally clever, or tried for long enough, we would eventually discover the pattern. To help us along, let's try the following technique: We'll divide each of our piles into subpiles in a certain way. Make a subpile of the largest power of two that you can. Then make a subpile of the next largest power of two, etc. Do this for each pile. For example, a pile of size three becomes subpiles of sizes 1 and 2. We'll use the notation (3) becomes (1,2). Similarly, (10) is (8,2), (15) is (8,4,2,1), (16) is (16), (27) is (16,8,2,1), etc.

Using this "subpile" idea, look for patterns that can help us determine a winning strategy in the following Nim games:

1. (0)(1)(2)(3) = (0)(1)(2)(1,2)
2. (1)(3)(4)(9) = (1)(1,2)(4)(1,8)
3. (2)(10)(15)(16) = (2)(2,8)(1,2,4,8)(16)
4. (1)(3)(4)(7) = (1)(1,2)(4)(1,2,4)
5. (2)(5)(6)(11) = (2)(1,4)(2,4)(1,2,8)

Do we see a winning strategy yet? If not, think about pairing off subpiles. Is it possible to make a winning move (i.e. take the last stone) if the subpiles are not all paired off? If they're not all paired off, what move can we make to pair them all off (and hence guarantee our opponent won't win on the next turn?)

Now we should be able to quickly determine whether the first or second player should win with perfect play in the following games

1. (2)(3)(7)(11)
2. (1)(2)(5)(9)
3. (4)(6)(9)(13)

Now, how about the following Chocolate games? Who wins?