

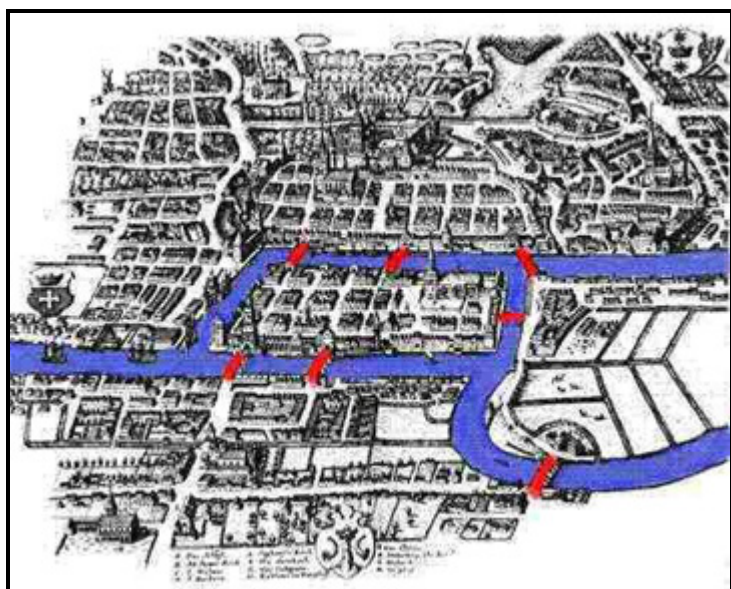
# Several Problems in Graph Theory

## The Seven Bridges of Königsberg

The Seven Bridges of Königsberg is a famous historical math problem that was finally solved by Leonhard Euler in 1735. By the end of today, you will be able to solve it as well.

Here is the problem:

The city of Königsberg, Prussia (now Kaliningrad, Russia) sat on both sides of the Pregel River and included two islands that were connected to both sides of the mainland and each other by seven bridges.



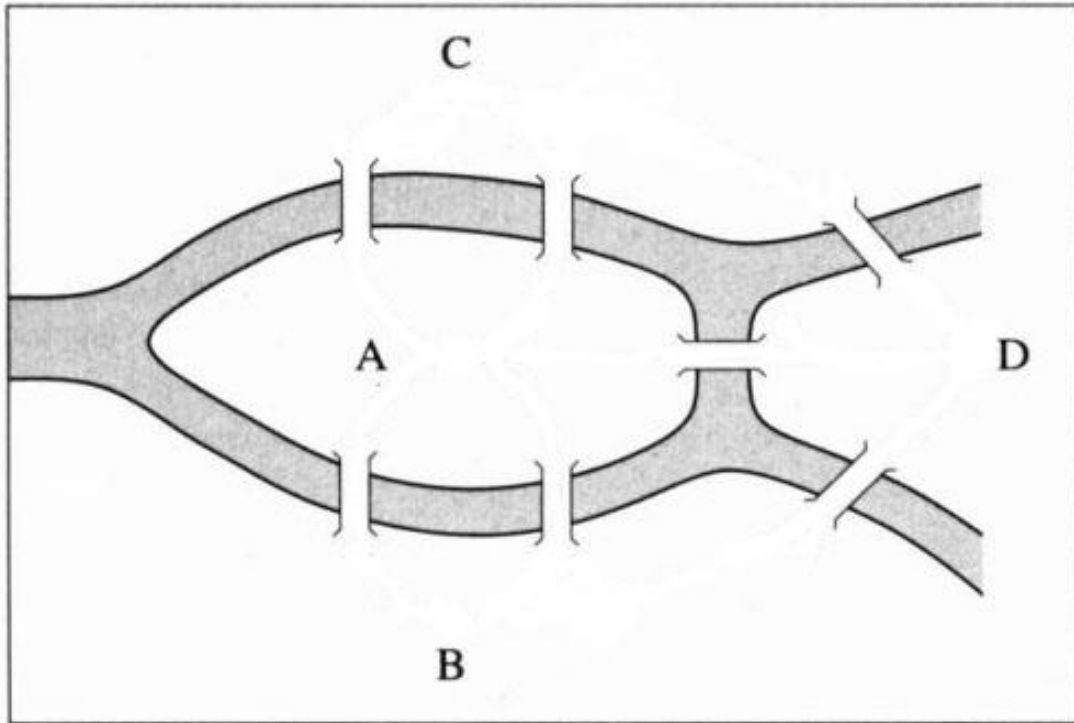
*Figure 1: A map of Königsberg from the 1700's with the seven bridges in red.*

The question is:

**Can a person find a path through the city that crosses each bridge exactly once?**

**Exercise 1:**

Consider this simplified picture of the city and its bridges. Try and find a path that crosses every bridge exactly once.



If you found such a path, draw it in above.

If you could not find such a path, list the any problems that you encountered when trying to find one:

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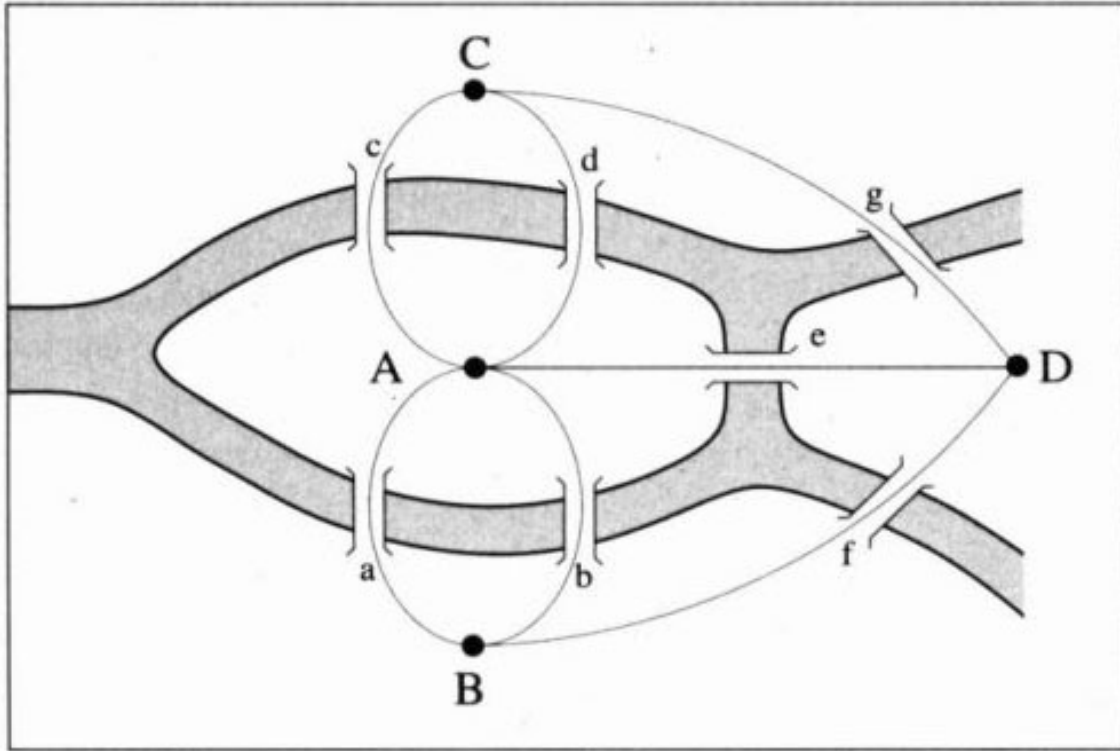
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Exercise 2)

Notice that the Königsberg Bridge picture can be redrawn with lines connecting the land areas A, B, C, and D.



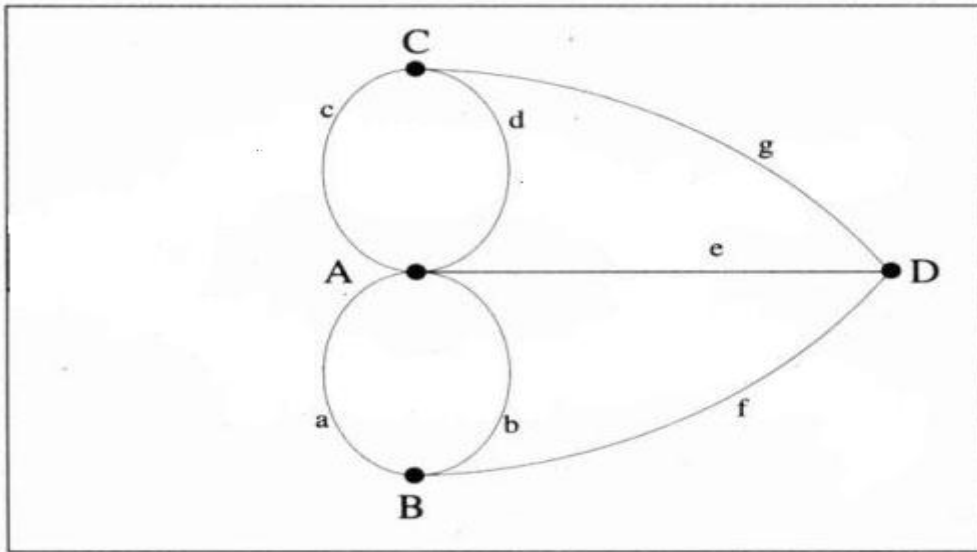
We have been trying to find a path that travels over each bridge exactly once.

Is this the same as trying to find a path that travels over every line exactly once?

Answer: \_\_\_\_\_

Explain: \_\_\_\_\_  
\_\_\_\_\_  
\_\_\_\_\_

Therefore, from now on, we can just consider the diagram:

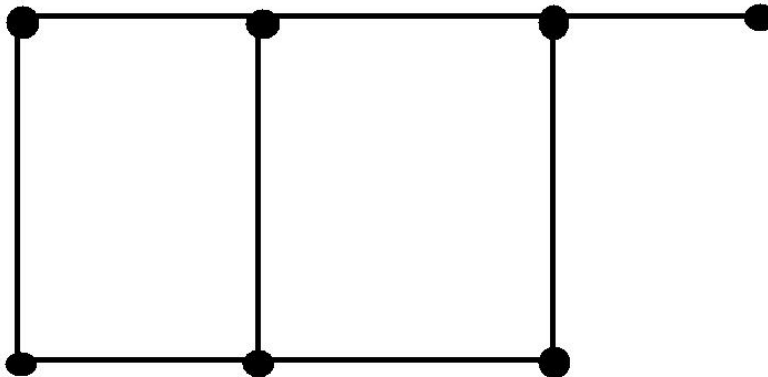


Notice that this figure is composed entirely of points connected by lines. Such a figure is called a **graph**.

The points are called **vertices** and the lines are called **edges**. Each edge has to have a vertex at each of its ends.

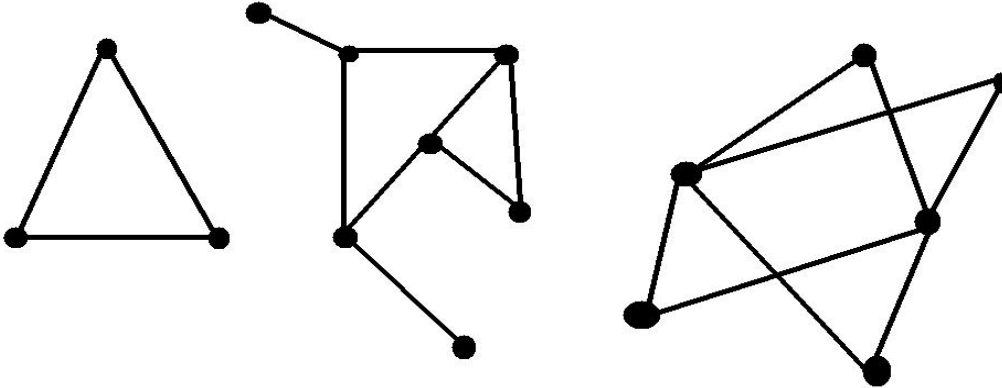
Sections that are enclosed entirely by lines are called **faces**. The above graph has exactly four faces.

For example, this is a graph with 7 vertices, 8 edges, and 2 faces. Label each of the faces, edges, and vertices:



Exercise 3)

a) Here are some other graphs:



How many vertices, edges, and faces does each of those three graphs have?

Graph 1: # vertices: \_\_\_\_\_ # edges: \_\_\_\_\_ # faces: \_\_\_\_\_

Graph 2: # vertices: \_\_\_\_\_ # edges: \_\_\_\_\_ # faces: \_\_\_\_\_

Graph 3: # vertices: \_\_\_\_\_ # edges: \_\_\_\_\_ # faces: \_\_\_\_\_

b) Get with a neighbor and draw graphs with the following properties:

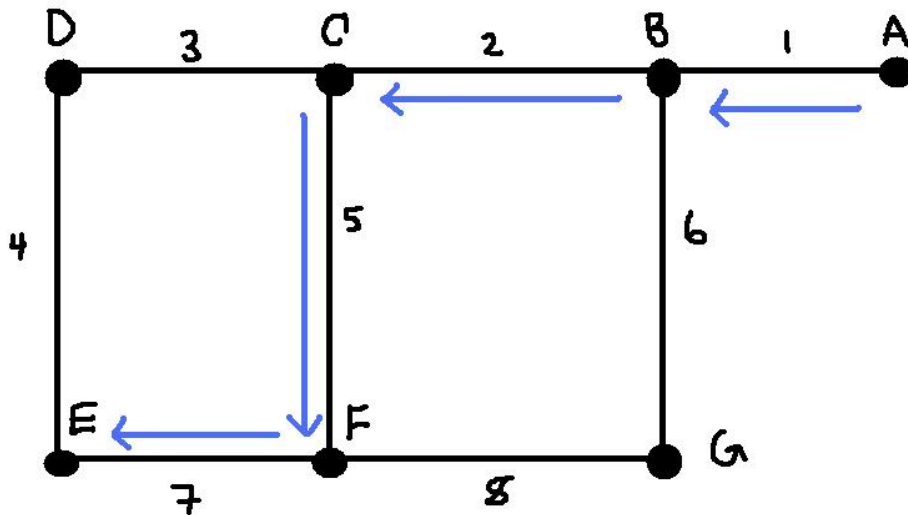
1) A graph with 5 edges and two vertices

2) A graph with 4 edges, 4 vertices, and 1 face

**Exercise 4)**

A **path** is just a way to travel from vertex to vertex along edges in a graph.

For example, the picture illustrates a path from vertex A to vertex E that crosses each of edge 1, 2, 5, and 7 exactly once.



Draw a path on the graph from vertex A to vertex C that crosses 6 edges exactly once. (You can work with a partner.)

Can you draw a path from C to D that crosses 5 edges? What about 6?

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Can you draw any paths that do not cross an edge twice and begin and end at the same point?

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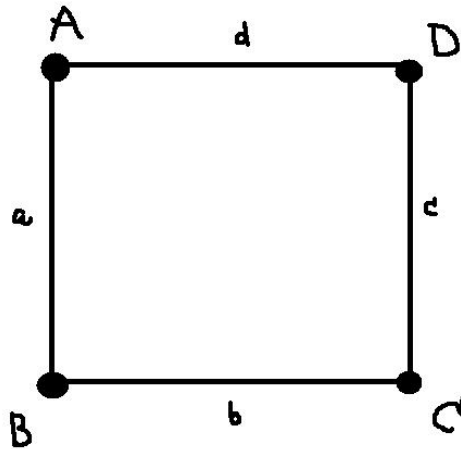
Can you draw a path that does not cross an edge twice and both begins and ends at vertex A?

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**Exercise 5)**

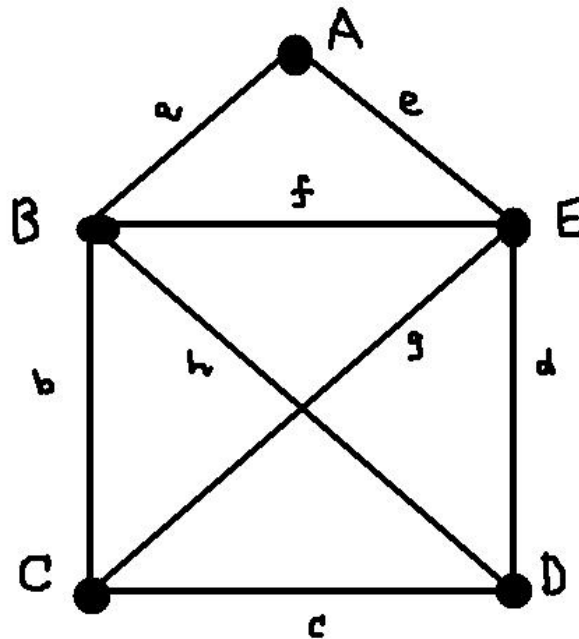
A graph has an **Eulerian path** if there exists a path that travels along each edge of the graph exactly once.

A) Draw an Eulerian path on the graph below that starts at vertex A:



It's easy to find Eulerian paths on graphs that trace out a polygon. What about for more complicated graphs?

B) Draw a Eulerian path on the following graph (all of the vertices and edges are labeled):

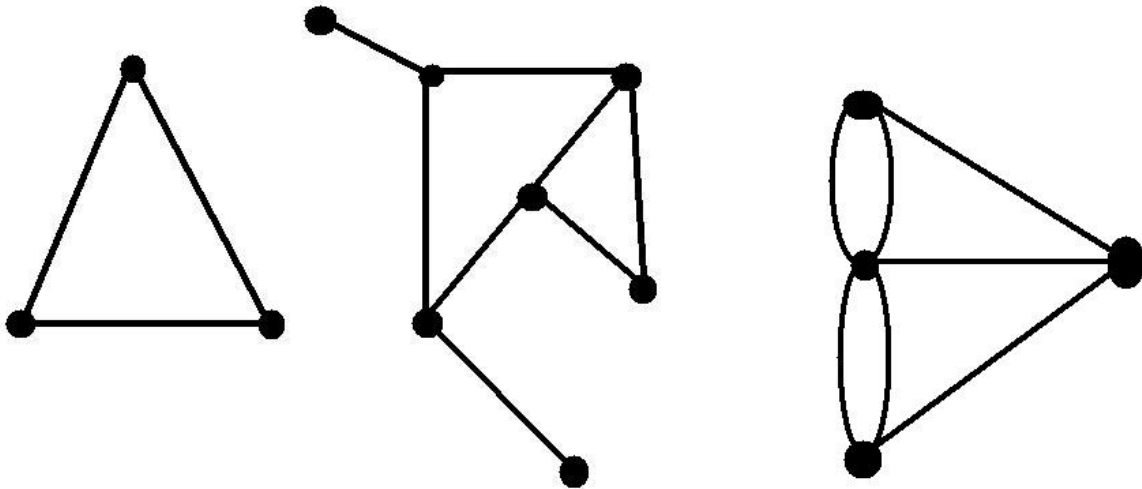


**Exercise 6)**

Before you investigate when a graph has a Eulerian path, you need one more definition.

The **degree** of a vertex is the total number of times that the vertex is the endpoint of an edge.

A) What is the degree of each vertex in the following graphs? (you can write the degree right next to the vertex)



How many odd and even vertices does each graph have?  
(An odd vertex has an odd degree and an even vertex has an even degree)

Graph 1: # odd vertices: \_\_\_\_\_ # even vertices: \_\_\_\_\_

Graph 2: # odd vertices: \_\_\_\_\_ # even vertices: \_\_\_\_\_

Graph 3: # odd vertices: \_\_\_\_\_ # even vertices: \_\_\_\_\_

B) Now, with a partner, draw graphs with vertices with the following degrees or explain why you cannot:

- a. A graph with 3 vertices, each of degree 2



- b. A graph with 4 vertices, 2 with degree 3 and 2 with degree 2
  
  
  
  
  
  
  
  
  
  
  
- c. Any graph that has only one vertex of even degree
  
  
  
  
  
  
  
  
  
  
  
- d. Any graph that has only one vertex of odd degree

Bonus: Can you draw a graph with exactly three vertices of odd degree? Draw or explain why you cannot:

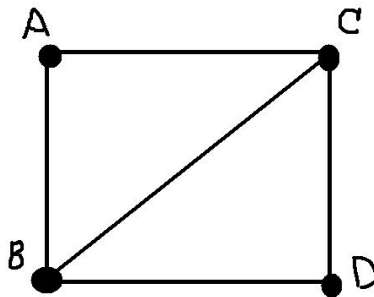
Can you draw a graph with three odd vertices? Four? Five? Seven? If you see a pattern on what can and cannot be drawn, write it down.

**Exercise 7)**

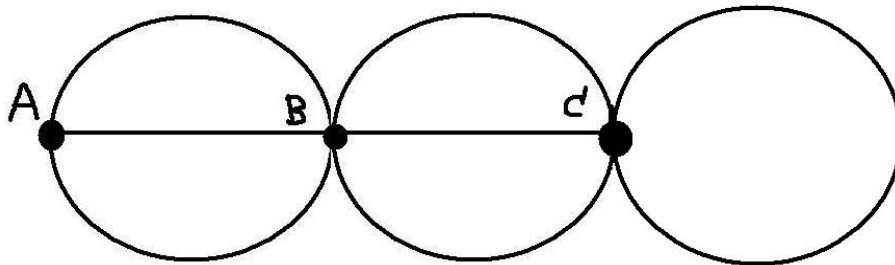
Now it's time to investigate when a graph has an Eulerian path. Consider the following examples and fill out the table.

Try and find an Eulerian path for all of the following graphs. If you can, draw it in. If not, explain in the margin what goes wrong.

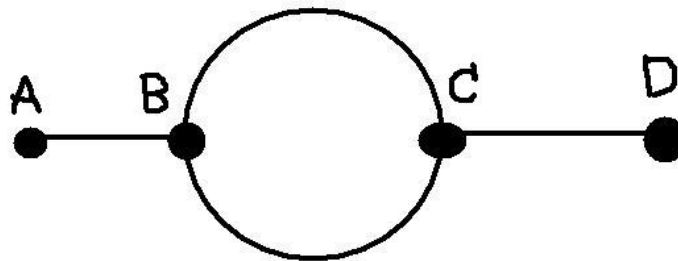
**Graph 1:**



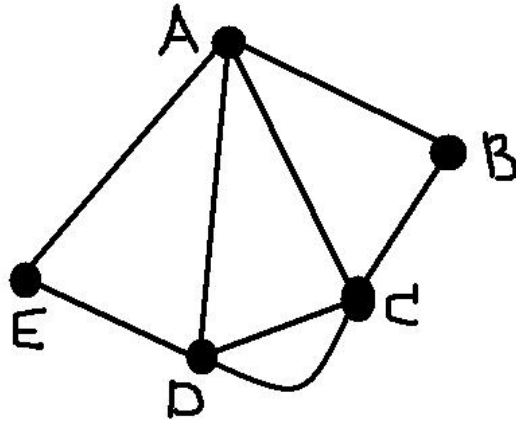
**Graph 2:**



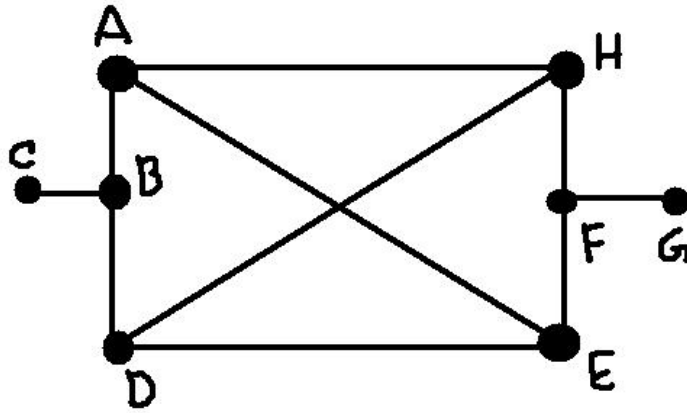
**Graph 3:**



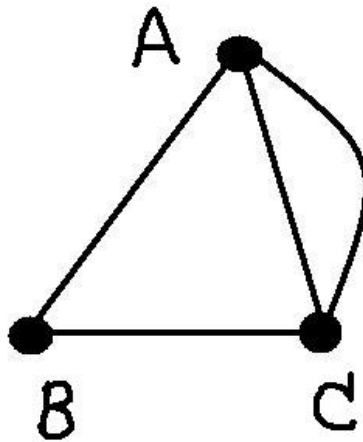
Graph 4:



Graph 5:



Graph 6:



Now, using graphs 1-6, fill in the following table:

	Graph 1	Graph 2	Graph 3	Graph 4	Graph 5	Graph 6
Did you find an Eulerian path?						
# of vertices with odd degree						
Degree of beginning vertex						
Degree of ending vertex						

Using the table, answer these questions with your partner.

If a graph had a Eulerian cycle, how many odd vertices did it have? (there are two possibilities)

In terms of the path, where did these odd vertices appear? Did they appear as the endpoints or in the middle of the path?

How many odd vertices do you think that a graph with an Eulerian path can have?

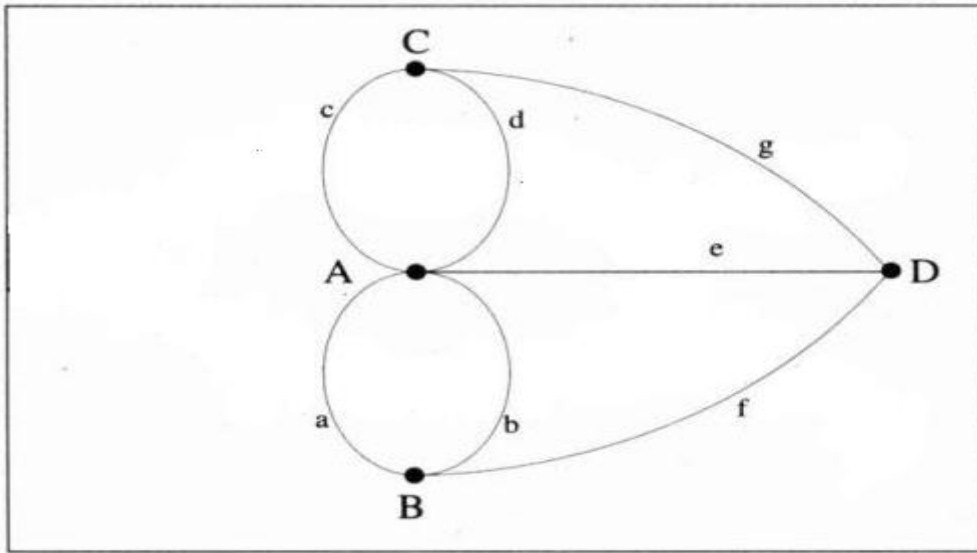
Make a conjecture about what characteristic will cause a graph to have (or not to have) an Eulerian path:

**Exercise 8:**

Lets Apply the result!!

In Exercise 7, you conjectured that a graph will have not have an Eulerian path if it does not have exactly zero or two odd vertices.

Let's consider the graph that represents the Bridges of Königsberg problem:



How many odd vertices does this graph have? \_\_\_\_\_

Using your previous result, does this graph have an Eulerian path? \_\_\_\_\_

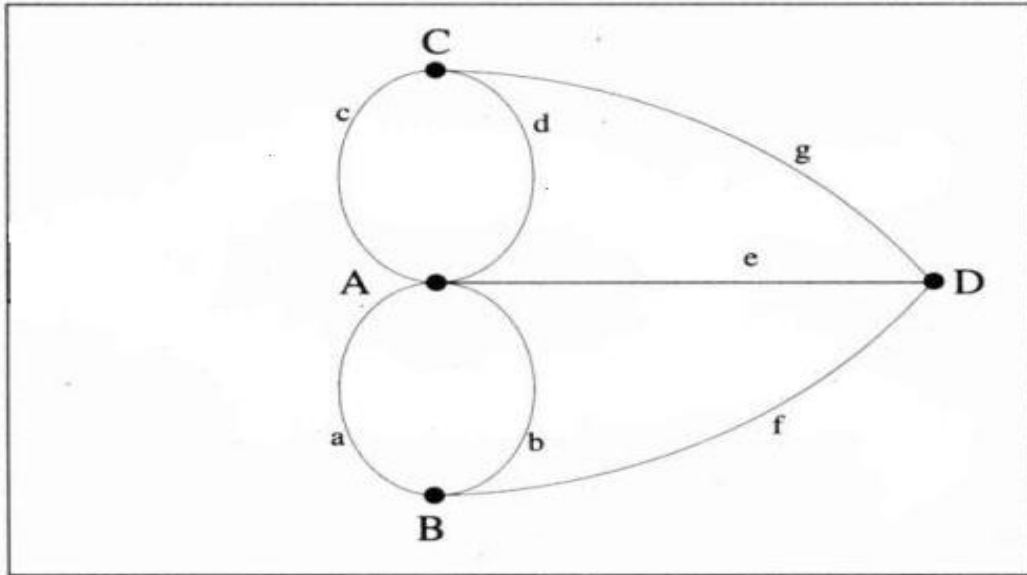
Now you can answer the big question:

**Can a person find a path through the Königsberg that crosses each of the seven bridges exactly once?**

**Answer:** \_\_\_\_\_

**Exercise 9:**

One more time, consider the Bridges of Koenigsburg. The graph shows that it is impossible to find a path that crosses each edge (bridge) exactly once.



If you add an edge to the graph (or a bridge to the city), can you find a path that crosses each edge exactly once?

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Draw the new edge on the graph and try and find a path that travels over each edge exactly once.

Is there more than one possible line you can draw to get such a path?

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Can you draw in two or more edges and still get such a path?

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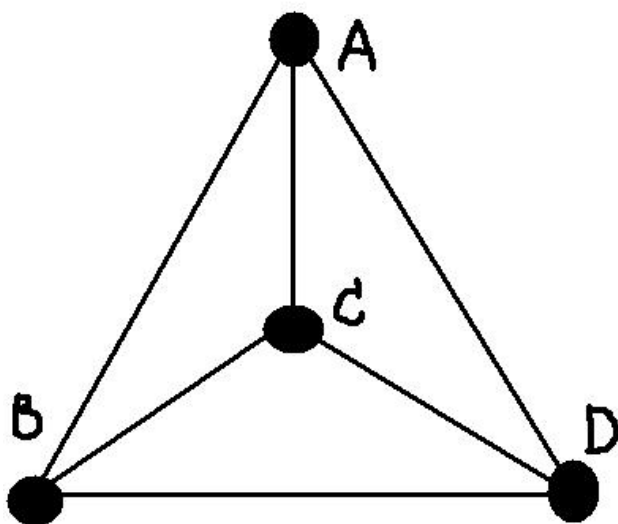
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**Exercise 10:                      Planar Graphs**

A graph that can be represented using vertices and edges that do not cross is called a **planar graph**; we say that the graph can be drawn in the plane.

Here is an example of a planar graph (notice that none of its edges cross):



If a planar graph is drawn so that none of its edges cross, there is a special relationship between the numbers of its edges, faces, and vertices.

On the next page there are two planar graphs already drawn. With a partner, draw three more. Write down the number of vertices, edges, and faces of each graph in table below:

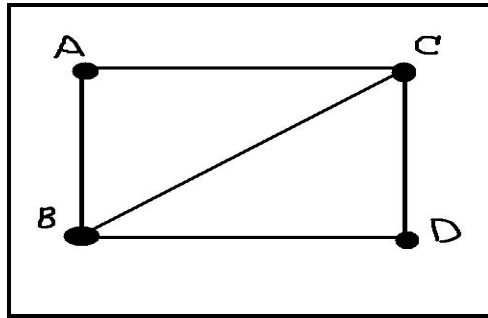
Graph	Number of Vertices (V)	Number of Edges (E)	Number of Faces (F)	V - E + F
1				
2				
3				
4				
5				

V-E+F should always be a specific number.

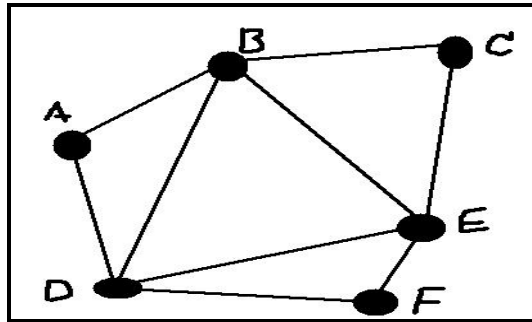
For any planar graph, what should V-E +F equal? \_\_\_\_\_

That is called **Euler's Equation**.

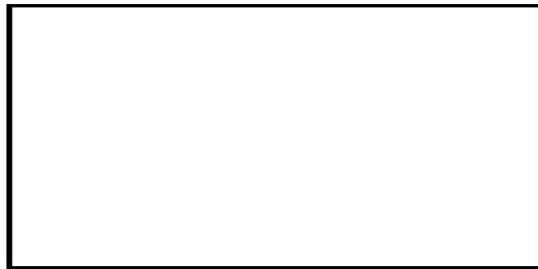
Planar Graph 1:



Planar Graph 2:



Planar Graph 3:



Planar Graph 4:



Planar Graph 5:



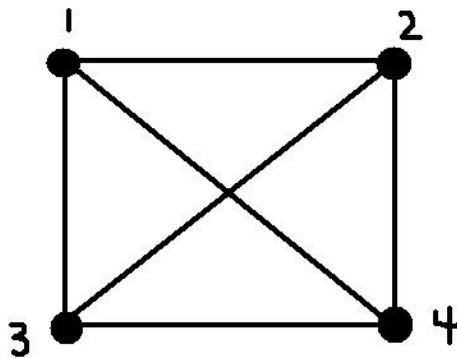


## Exercise 11)

## Complete graphs

A **complete graph** is a graph where every pair of vertices is connected by exactly one edge. An  **$n$ -complete graph** (also known as a **complete graph on  $n$  vertices**) is a complete graph with  $n$  vertices.

For example, here is the 4-complete graph:



Draw the 1,2,3, 5, and 6- complete graphs in the boxes below:

1-complete graph:



2-complete graph



3-complete graph:



5-complete graph:



6-complete graph:



One interesting question is:

**How many edges does an  $n$ -complete graph have?**

Fill out the following table:

$n$	Number of Edges in an $n$ -complete graph	Number of Edges in an $(n-1)$ -complete graph	(Number of Edges in an $n$ -complete graph) - (Number of edges in an $(n-1)$ -complete graph)
1		-	-
2			
3			
4			
5			
6			

**Make a conjecture:**

Look at the table. Do you see a pattern between the number of edges in an  $(n-1)$ -complete graph and the number of edges in an  $n$ -complete graph? What is it?

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If we know that an  $(n-1)$ -complete graph has  $m$  edges, how many edges does an  $n$ -complete graph have?

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## How many edges does an $n$ -complete graph have? (cont.)

Now consider any  **$n$ -complete graph** and Choose any vertex on it.

How many edges have ends that connect to that vertex? Why? (If you are not sure, look back at the  $n$ -graphs we have already drawn and try to find a pattern)

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This graph has  $n$  total vertices. Each one has  $(n-1)$  edge ends connected to it. How many total edge ends are there in the graph?

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How many edge-ends does each edge in the graph have? (hint: there is a starting end and an ending-end)

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How many edges are in the  $n$ -complete graph?

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