Olympiad
High School

WU Math Circle

April 25, 2010

1. For how many integers x in the set $\{1, 2, 3, \ldots, 99, 100\}$ is $x^3 - x^2$ the square of an integer?

2. What is the coefficient of x^{50} in the expansion of the following product?

$$(1 + 2x + 3x^2 + 4x^3 + \cdots + 101x^{100})(1 + x + x^2 + x^3 + \cdots + x^{25})$$

3. What are the real roots of $x^3 - x - 6$?

4. If $i^2 = -1$, then what is the value of i^{2010} ?

5. Write x, y and z in order from largest to smallest if $x = 2^{100}$, $y = 3^{75}$ and $z = 5^{50}$.

6. What is the coefficient of x^9 in the expansion of

$$\left(\frac{x^2}{5} - \frac{25}{x} \right)^9 ?$$

7. How many real roots does $x^5 - 16x$ have?

8. Find the solutions of the equation $\log_3(a^2 - 15a) = 2$.

9. For what value of b is $\frac{3i+2}{bi-2}$ a real number (here $i^2 = -1$)?

10. If $p = \frac{s}{(1+k)^n}$, then solve for n.

11. Let θ be an acute angle such that $8cos(2\theta) + 8sec(2\theta) = 65$. Find the value of $4cos(4\theta)$.

12. If $x - y = 13$ and $x^2 - y^2 = 169$, then what is xy?

13. Find sum of the minimum distance and the maximum distance from the point $(8, 1)$ to the circle $x^2 + y^2 + 6x - 8y - 11 = 0$.

14. If we are cutting a cake and Bob gets twice as much as Mike who gets twice as much as Alice who gets twice as much as Audrey who gets twice as much as Tim, what fraction of cake does Mike get?

15. Determine the equation of the line passing through the points of intersection of the circles $(x+2)^2 + (y+1)^2 = 9$ and $(x+3)^2 + (y+4)^2 = 16$.

16. A regular hexagon has sides 8in long. Find the area of the triangle formed by connecting alternate vertices.

17. Hannah bought a box in the shape of a rectangular prism. She found the lengths of the diagonals of the faces to be $20, 2\sqrt{61}, 2\sqrt{89}$. Determine the volume of the box.

18. In value, the area of an equilateral triangle is equal to its perimeter. What is the length of the side of the triangle?

19. We have a cone whose height is 10in and has radius 6in at its opening. How much water is in the cone when the water reaches a height of 5in from the tip at the bottom?

20. Find the real solutions to $4^{x^2-3} = 2^{x^2-2x}$.

21. Where does the circle $x^2 + y^2 = 9$ intersect the hyperbola $x^2 - y^2 = 1$?

22. A line passes through $(1, 2)$ and $(-4, -3)$. Find the equation of the line perpendicular to that line and passing through $(0, 0)$.

23. A cube and sphere have the same surface area. What is the ratio of the volume of the sphere to that of the cube?

24. Let $f(x)$ be a function such that, for every real number x, $f(x) + 2f(x) = \cos x$. What is the value $f\left(\frac{\pi}{4}\right)$?

25. Find all the real solutions to $x^6 - 1 = 0$.

26. If $x + \frac{1}{x} = 5$ and $x > 1$, what is the value of $x^3 - = \frac{1}{x^3}$?

27. Find a polynomial $f(x)$ that satisfies $(x-1)f(x)-xf(x-1) = x^2-x-1$.

28. How many unique lists of length 4 can you make which comprise only the numbers $1, 2, 3$? ($1, 2, 2, 3$ is different from $2, 1, 2, 3$)

29. What is the value of $1 + 2 + 3 + \cdots + 99 + 100$?

30. What is the value of $1 + 2 + 4 + 8 + \cdots + 512 + 1024$?