Compass and Straightedge Constructions

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This time we are going to do some geometry and drawings.

As hinted by the title, we only need a compass and a straightedge, that is, a ruler without measures.

Formally stating, the regulations for our tools are:

> The **compass** can be opened arbitrarily wide, but it has no markings on it. It can only be opened to widths that have already been constructed.
>
> The **straightedge** is infinitely long, but it has no markings on it and has only one edge. It can only be used to draw a line segment between two points or to extend an existing line.

For today's adventure we are going to explore the constructions to see what we can do, from some easy practice to some challenging problems.

1 Level One

Problem 1.1 *Mark two points A and B below. Then draw the line through them and then find the segment $\overline{AB}$. (Recall that $\overline{AB}$ means the line segment between points A and B.)*

Problem 1.2 *Mark a point C below. Then draw the circle that centers at C and the radius $r = AB$, the length of the $\overline{AB}$ from the previous problem.*

Problem 1.3 *Mark a point D. Draw a line segment $\overline{DE}$ such that $CD = AB$. Again, AB is the line segment you get from the first problem.*

Problem 1.4 *Mark three points A_1, A_2 and A_3 and make sure that they are not on the same line. First draw the angle $\angle A_1A_2A_3$ and then draw the triangle $\triangle A_1A_2A_3$.*

These are basic constructions. If you are done, you can begin with the next level.

2 Level Two

Problem 2.1 *Mark two points A and B. Draw the line l through them and find another point C on l such that $AB = BC$.*

Problem 2.2 *Mark two points A and B. Draw an equilateral triangle above the segment $\overline{AB}$.*

Problem 2.3 *Mark two points A and B. Draw the mid-perpendicular of the segment $\overline{AB}$, and find the middle point of $\overline{AB}$.*

Problem 2.4 *Draw a line l and a point A not lying on l. Draw the line m that is passing through A and perpendicular to l.*

Problem 2.5 *Draw a line l and a point A not lying on l. Draw the line m that is passing through A and parallel to l.*

When you are done, please go the the next level :)

3 Level Three

Problem 3.1 *Draw an angle* $\angle ABC$*. Find the bisector of this angle.*

Problem 3.2 *Draw a line* l *and two points* A *and* B *not lying on* l*. Make sure that the line* $\overleftrightarrow{AB}$ *is not perpendicular to* l*. Find the point* C *on* l *such that* $AC = BC$*. Hint: consider the mid-perpendicular of* $\overline{AB}$*.*

Problem 3.3 *Draw three angles such that* $\angle A = 90^\circ$, $\angle B = 60^\circ$ *and* $\angle C = 30^\circ$.

Problem 3.4 *Draw a random triangle. Then draw another one that is congruent it. Hint: if three pairs of sides of two triangles are equal in length, then the triangles are congruent.*

Problem 3.5 *Draw a random angle. Then draw another one that is congruent it. Hint: try constructing congruent triangles.*

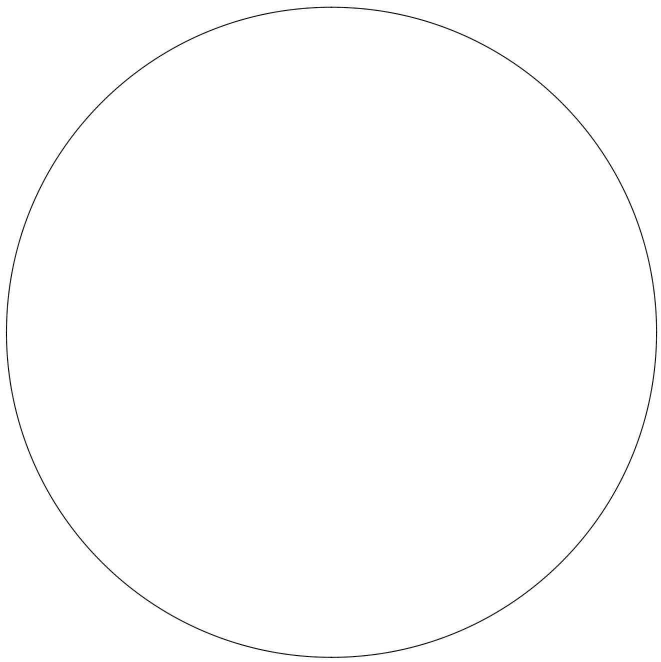
Problem 3.6 *Draw a line segement $\overline{AB}$, and then draw an equilateral triangle with the center at A and a vertex at B.*

Problem 3.7 *Draw a line l and a point A not lying on l. Then draw the circle that centers at A and tangent to l.*

Problem 3.8 *Mark two points A and B. Draw a circle that centers at A and does not pass through B. Then draw another circle that centers at B and is tangent to the previous circle.*
And that is the end of this level. From next level, problems will be more challenging.

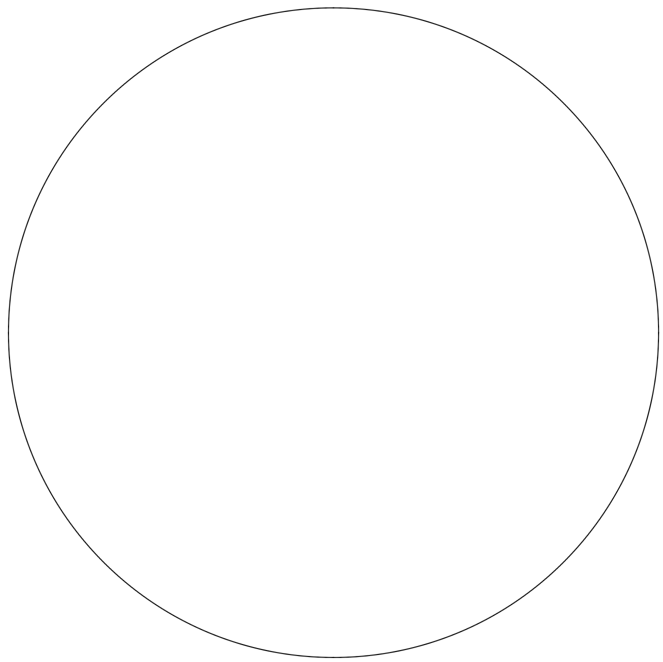
4 Level Four

Problem 4.1 *Find a diameter of the circle below.*

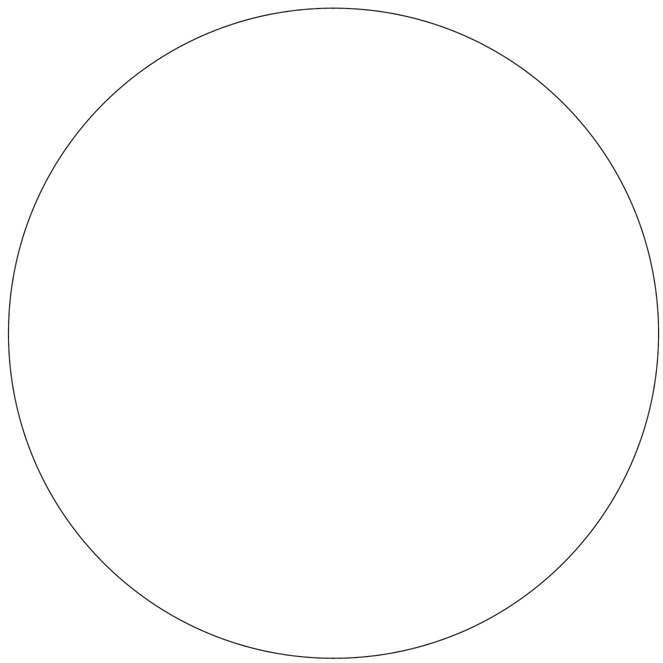


Problem 4.2 *Find the center of the circle above.*

Problem 4.3 *Draw a equilateral that has all its vertices on the circle below. Also draw a regular hexagon inscribed in the circle.*



Problem 4.4 *Draw a square that is inscribed in the circle below.*



Problem 4.5 *Construct a regular pentagon, following the hints:*

1. Mark a point O, draw a circle centering at O.

2. Draw a diameter. It touches the circle at points A and B. Of course it passes through the point O.

3. Draw a line segement $\overline{OC}$ with end points O, the center of the circle, and C, a point on the circle, such that $\overline{OC}$ is perpendicular to $\overline{AB}$.

4. Find the middle point D of the line segment $\overline{OC}$.

5. Draw the bisector $\overline{DE}$ of the angle $\angle ADO$, which intersects $\overline{AO}$ at the point E.

6. Draw a line that is parallel to $\overline{CO}$ and passing through E. It intersects the circle at such a point that is on the same side of $\overline{AB}$ with respect to the point C. Denote this point A_1.

7. $\overline{AA_1}$ is one edge of the regular pentagon. With the help of this edge and the circle, you can find the other vertices of the pentagon: A_2, A_3 and A_4, and finally you get the regular pentagon.

Problem 4.6 *Draw a random line segment and then trisect it. Hint: make this segment into a triangle and construct similar triangles.*

5 Level Five

There are some other interesting challenges:

Problem 5.1 *A point C is lying outside a line $\overleftrightarrow{AB}$. Construct a point D symmetric to C with respect to $\overleftrightarrow{AB}$, using only a compass*

Problem 5.2 ***NAPOLEON'S PROBLEM***
Given a circle and its center, divide this circle into four equal arcs, using only a compass.

This one is constructible but complicated. You can try it if you have interest :)

6 MISSION IMPOSSIBLE

The following three problems of compass and straightedge constructions had been troubling mathematicians for about 2000 years.

Problem 6.1 ***SQUARING THE CIRCLE***
Construct a square with the same area as a given circle.

Problem 6.2 ***DOUBLING THE CUBE***
Construct the side of a cube that has twice the volume of a cube with a given side.

Problem 6.3 ***ANGLE TRISECTION***
Construct an angle tha is one-thrid of a given arbitrary angle.

They are proved to be impossible.

There are some other constructions that cannot be done soly relying on a compass and a straightedge, including: a regular polygon with 7 sides, 9 sides, etc.