Mersenne Primes

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Problem 1

1. Write out the number $2^n - 1$ for $n = 1, 2, 3, 4, 5, 6, 7, 8,.$
2. Factor each of these numbers.

3. Are they all prime? What kind of pattern can we guess for $2^n - 1$ to be prime?

4. In 1536, Hudalricus Regius showed that $2^{11} - 1 = 2047$ was not prime. Find all prime divisors of 2047.

   By 1603, Pietro Cataldi had correctly verified that $2^{17} - 1$ and $2^{19} - 1$ were both prime, but then incorrectly stated $2^n - 1$ was also prime for 23, 29, 31 and 37. In 1640 Fermat showed Cataldi was wrong about 23 and 37; then Euler in 1738 showed Cataldi was also wrong about 29. Sometime later Euler showed Cataldi's assertion about 31 was correct.

5. Find a prime divisor of $2^{23} - 1$, and $2^{37} - 1$. (Hint: If p is an odd prime, all prime divisors of $2^p - 1$ have the form $2kp + 1$.)

   French monk Marin Mersenne (1588-1648) stated in the preface to his *Cogitata Physica-Mathematica* (1644) that the numbers $2^n - 1$ were prime for

   $$n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 \textit{ and } 257, \qquad \text{for } n < 257.$$

   Unfortunately, this statement is wrong! By 1947 Mersenne's range, $n < 258$, had been completely checked and it was determined that the correct list is:

   $$n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107 \textit{ and } 127.$$

   As of October 2009, only 47 Mersenne primes are known. The largest known prime number ($2^{43,112,609} - 1$) is a Mersenne prime.

Problem 2

1. Check that 6 is the smallest number that is equal to the sum of all of its positive divisors, excluding itself. (Note: Such a number is called a perfect number)
2. Find the next perfect number.

3. Factor the first 2 perfect numbers, and also the next two, 496 and 8128.
4. For each of the above four perfect numbers, what are the odd primes for that number? Are they Mersenne primes?
5. Is there a pattern of perfect numbers?

We have the following theorem.

Theorem: k is an even perfect number if and only if it has the form $2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

So the search for Mersennes is also the search for even perfect numbers!

6. Find the next 4 perfect numbers. (Hint: Use the above Theorem.)

Problem 3

1. We can write $1 = 2^0, 2 = 2^1, 3 = 2^1 + 2^0, 4 = 2^2, 5 = 2^2 + 1$. Write 5,6, 7, 8, 9, 10 into a sum of distinct powers of 2?

2. In binary, we write 1=1, 2=10, 3=11, 4=100, 5=101, 6=110, 7=111, 8=1000. Write 9, $\cdots$, 32 into binary.

3. Write all Mersenne numbers $2^n - 1$ for $n \leq 32$ into binary.
4. Write all perfect numbers less than 10000 into binary.

There are many interesting open questions. For example,

1. Find an odd perfect number?
2. Find all Mersenne primes?

References:

1. $http://primes.utm.edu/mersenne/index.html\sharp known$
2. $http://en.wikipedia.org/wiki/Mersenne\_prime\sharp Perfect_numbers$