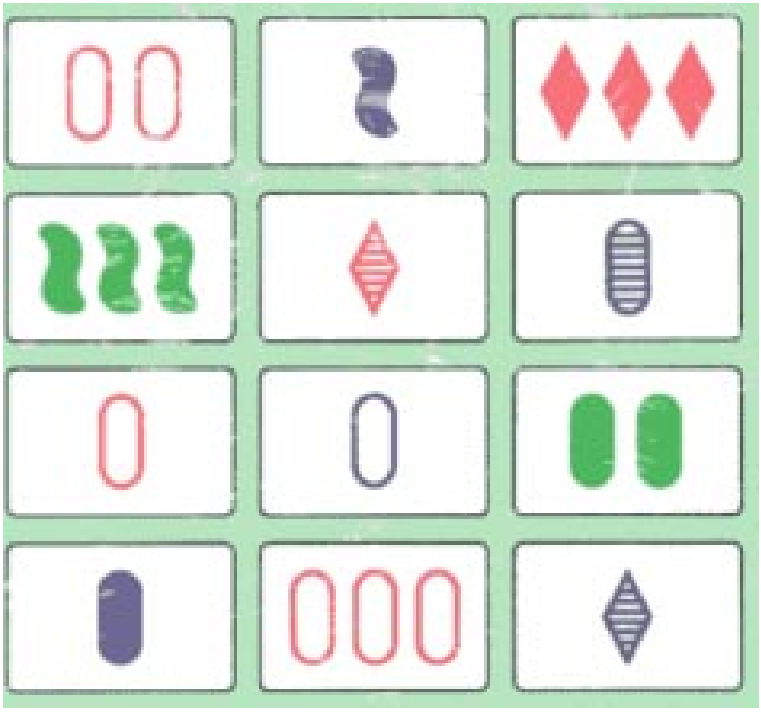
The Game of Set®



Andrew Womack

October 3, 2010

1 The Set®Deck

Set is a card game where each card has four attributes: color, number, shape, and shading. There are three possibilities for each attribute: color- red, green, purple; number- one, two, three; shape- oval, diamond, squiggle; shading- open (outline), striped, solid (filled). There is one of each card for any choice of the four properties. So there is only one card that looks like $(green, 3, diamond, striped)$. A set consists of any three cards with the following property: for any given attribute, the cards have the same value or all the card have different values. In a set, the ordering of the cards does not matter.

Q1: How many cards are in a set deck?

Q2: If two cards are red, what color does a third card need to be in order to complete a set?

Q3: If one card is striped and another card is solid, what shading does a third card need in order to complete a set?

Q4: Given any two cards, how many card choices are there for completing a set?

Q5: Given the card $(green, 3, diamond, striped)$ and a second card that you know is green, diamond, and striped; how many sets can you make?
Q6: Given the card $(green, 3, diamond, striped)$ and a second card that you know is red and 3; how many sets can you make?
Q7: How many sets are there in a Set®Deck?
Q8: Given the card $(purple, 1, squiggle, solid)$, to how many sets does it belong?
Q9: Does the number of sets to which a card belongs depend on the card?
Q10: Given two sets of cards, how many cards can those sets have in common?
Q11: If you have 6 Set®cards, what is the maximum number of sets you can make?
Q12: If you have 9 Set®cards, what is the maximum number of sets you can make?
Q13: Can you make a 3×3 Set®magic square? Here each row, column, and diagonal (even off main) must be a set.
Q14: What is the smallest collection of cards so that every card in the collection must have a set in the collection?

2 A Binary operation on Set®Cards

We can represent each set card as a list $(color, number, shape, shading)$. We can call these lists by variable names a, b, c, d, x, y, z. For any two cards x, y define their product $x \star y$ to be the card that completes a set with x and y.
Q15: Is $\star$ commutative?
Q16: Is $\star$ associative?
Q17: Does $x \star x$ make sense? How would you define it?
Q18: Can you define an analog of $\star$ on each attribute?
Q19: What is $x \star (x \star y)$?
Q20: Complete the multiplication table on $\{1, 2, 3\}$

$\star$	1	2	3
1			
2			
3			

Q21: Can you express $\star$ in terms of elementary operations? Elementary operations to use are the the sum or product of any two numbers, multiplication by any integer, and taking remainders after dividing by any integer.
Q22: Identifying 3 with 0 makes $\star$ a multiplication on the integers modulo 3 (remainders after dividing by 3). Does 0 behave in the usual way? Does 1 behave in the usual way?
Q23: Define identities between the atrtibutes and the integers modulo 3:

$$(diamond, squiggle, oval) \to (\_\_, \_\_, \_\_)$$
$$(open, striped, full) \to (\_\_, \_\_, \_\_)$$
$$(red, green, purple) \to (\_\_, \_\_, \_\_)$$
$$(1, 2, 3) \to (\_\_, \_\_, \_\_)$$

Q24: What card is represented by $(0, 2, 1, 0)$?
Q25: Find the following products:

$$(1, 0, 0, 0) \star (1, 0, 0, 1)$$
$$(0, 1, 0, 0) \star (0, 2, 0, 0)$$
$$(2, 2, 2, 2) \star (0, 2, 2, 1)$$
$$(0, 0, 0, 0) \star (0, 1, 2, 1)$$

Q26: The numbers $\{0, 1, 2, \ldots, 80\}$ can be represented as trinary numbers (base 3) with less than 5 digits since $81 = 3^4$. Can you find a natural relationship between the 4-tuples and $\{0, 1, 2, \ldots, 80\}$?
Q27: What card is represented by the base 10 number 75?
Q28: How is $\star$ defined on the base 10 integers modulo 81?
Q29: What is the product $37 \star 42$ if 37 and 42 are base 10 numbers?

3 A Game of Set®

12 Set®cards are laid face-up on a table. When a participant sees a set, she or he calls out set and then removes the set from the table. If no set is seen, then the participants add three new cards to the table. Whenever there are less than 12 cards on the table, cards are added until there are 12 cards. The game ends when all of the sets are exhausted from the deck. The person with the most sets wins. **Think about the game with a smaller number of attributes first.**
Q30: Is it possible that there are no sets in 12 cards?
Q31: Is it possible that there are no sets in 15 cards?
Q32: Is it possible that there are no sets in 18 cards?
Q33: Is it possible that there are no sets in 21 cards?
Q34: What is the largest collection of cards that contains no sets?
Q35: How many cards can be left on the table at the end of a game of set?
Q36: Can a game end in a tie?

4 Extensions

Invent the game of set with more than 4 attributes (for instance add the attribute *size*). Answer all of the previous questions with this added attribute. Some of the questions are straightforward, some are easy, and some are insanely hard. What if you were to have 4 attributes each with 4 choices and required sets to contain 4 cards?