CRISS-CROSS APPLESAUCE!

On the next page is a triangular game-board. *Don't* draw the edges yet. *Do* draw up to seven big dots *inside* the triangle (not too close together).

Then find someone to play the game with. You make the first move on your own game board. On each turn, draw a segment between any pair of dots that doesn't intersect any of the segments already drawn. The winner is the last person able to make a legal move.

When you've finished playing fill out the rest of this sheet.

(1) How many dots did you draw?

Who won your game?

(2) Count the following on your board. Don't forget to count the outside of the board as a face!

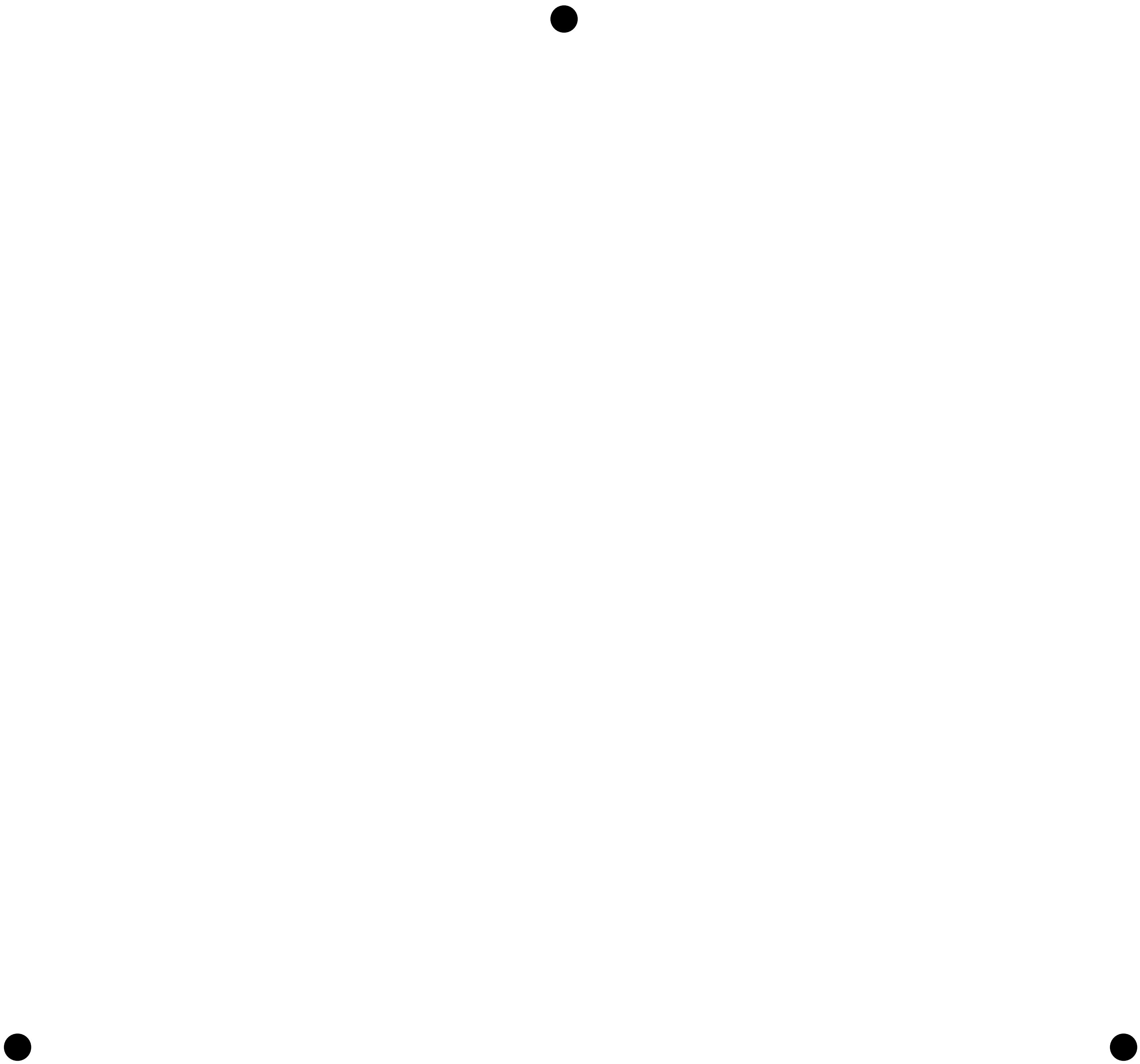
vertices (points)	edges (segments)	faces (regions)
V=	E=	F=

(3) We'll share everyone's results now.

(a) What do you notice about which of your classmates won their own games?

(b) What about the numbers of vertices, edges and faces?

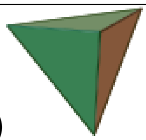
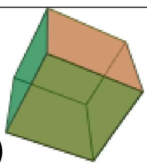
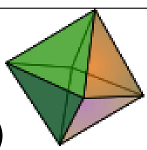
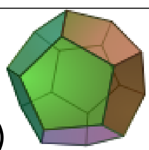
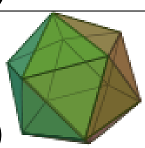
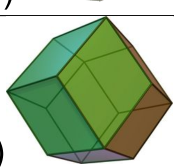
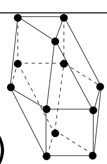
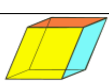
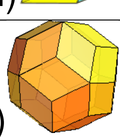
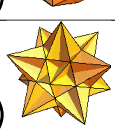
Game Board



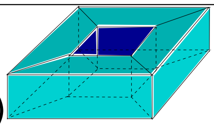
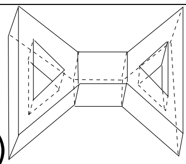
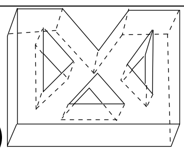
Polytoys!

There are polytopes scattered around the classroom, and labeled by roman numeral. Fill in the table for several of them below. Then try to figure out one or two on the next page. Make use of any patterns or symmetries you can to shorten the computation of A, the *sum of all the angles* on the outside of a polytope.

(1) Polytopes without holes:

	name	$V =$ (vertices)	$E =$ (edges)	$F =$ (faces)	$A =$ (sum of angles)
(i) 					
(ii) 					
(iii) 					
(iv) 					
(v) 					
(vi) 					
(vii) 					
(viii) 					
(ix) 					
(x) 					

(2) Polytopes with holes: only one of these has a model. For the others, you'll have to use your imagination together with the bigger pictures on the next page. You may assume all angles are a multiple of $45°$.

	name	$V =$ (vertices)	$E =$ (edges)	$F =$ (faces)	$A =$ (sum of angles)	$G =$ (# of holes)
(xi) 	donut					
(xii) 	bow tie					
(xiii) 	pretzel					

[Note: "G" stands for *genus*.]

Do these data repeat the pattern you found above, or are they different (and how)?

(3) If you got through all that, why not try **higher dimension**?

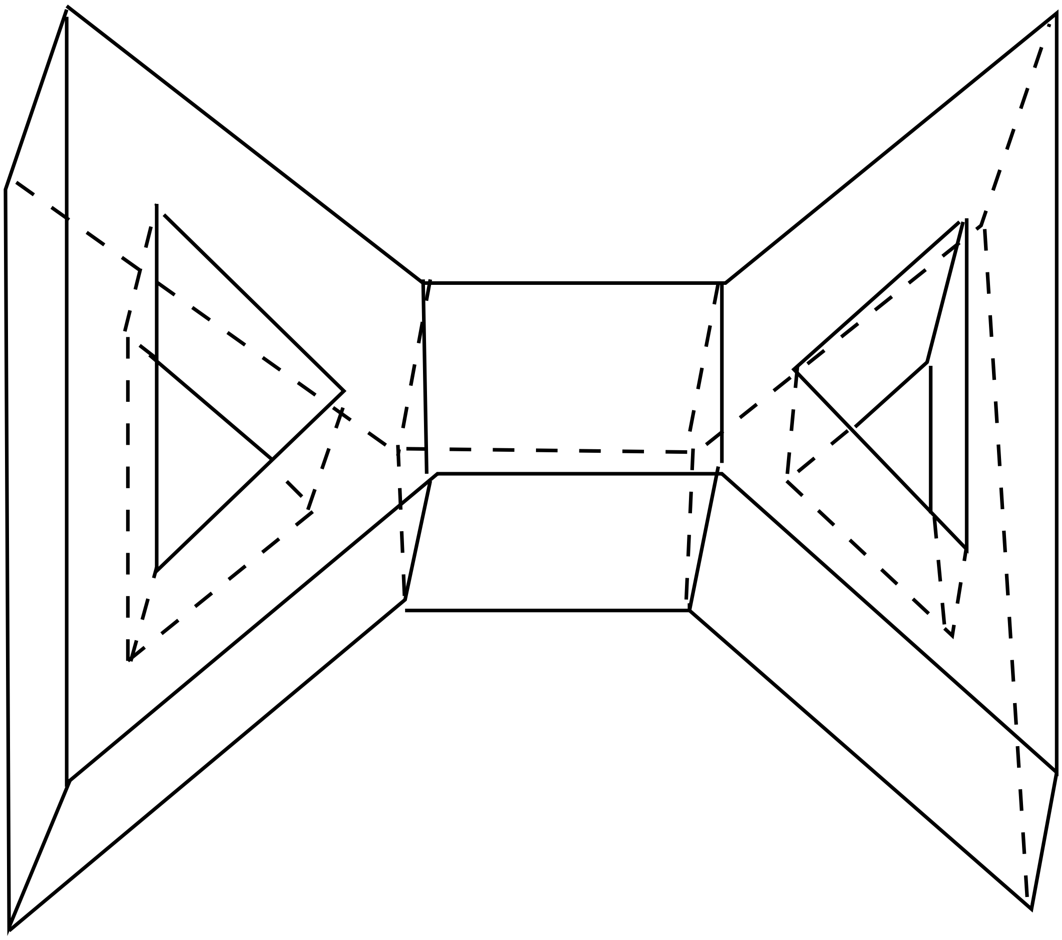
- A hypercube is a 4-D version of a cube – in (x, y, z, w)-space, it takes up the region where all four coordinates are between 0 and 1. It has a new feature: "3-D faces" called hyperfaces.
- A pentahedron is a 4-D version of the tetrahedron – in (x, y, z, w)-space, it takes up the region where all four coordinates are positive and the sum $x + y + z + w \leq 1$. (Its 3-D cross-sections are tetrahedra!)

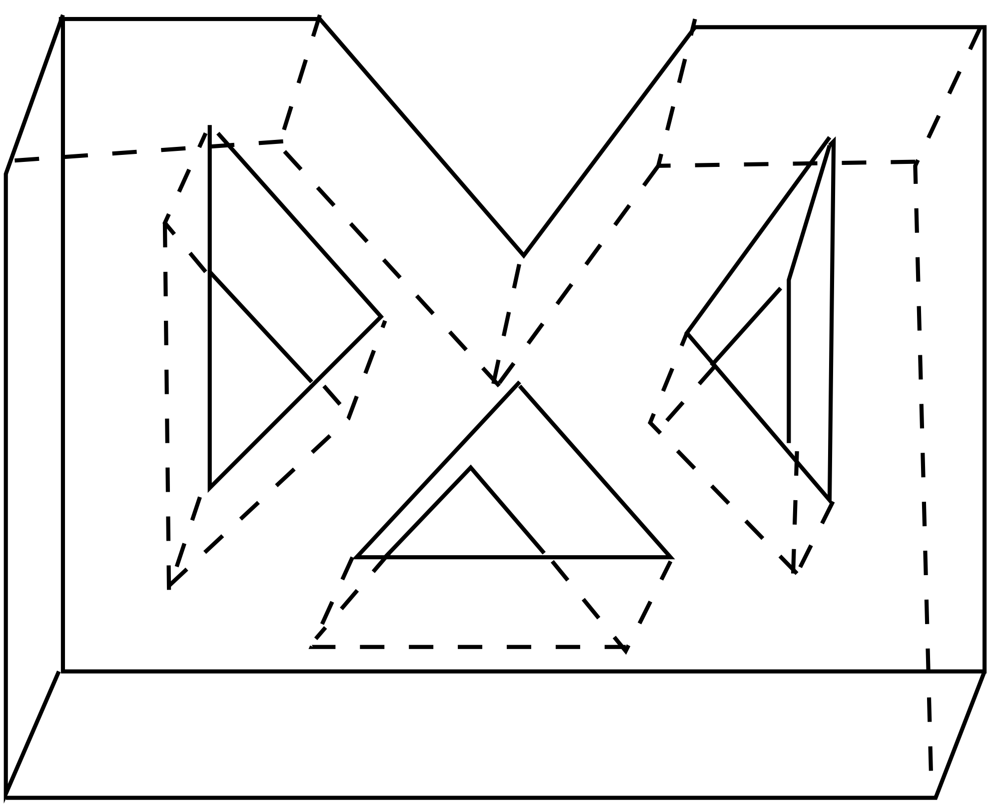
	$V =$ (vertices)	$E =$ (edges)	$F =$ (2-D faces)	$H =$ (3-D faces)
hypercube				
pentahedron				

Notice anything relating the two sets of numbers?

Can you give a formula for the number of vertices, edges, etc. for an "n-dimensional cube"?

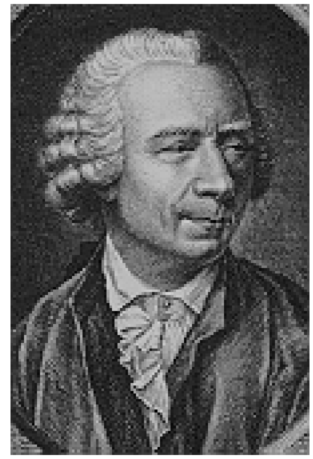
(Larger pictures, for counting vertices, edges, etc.)





EULER SPOILER

For a three-dimensional polytope, the Euler characteristic χ is defined by:



$$\chi = V - E + F.$$

What values of χ do you get from your data for the polytopes without holes?

What about the ones *with* holes? (Remember G is the number of holes.)

Try to guess a formula relating χ and G. We'll use this (just when $G = 0$) on the next two pages.

In higher dimension, $\chi = V - E + F - H + \cdots$. What do you get for the hypercube and pentahedron?

The "n-cube"?

Back to criss-cross!

Let N denote the number of points you added, and remember

$$\begin{aligned} V &= \text{number of vertices} = N+3 \\ E &= \text{number of edges} \\ F &= \text{number of faces} \end{aligned}$$

The data from our criss-cross game suggested three things:

(a) $3F = 2E$
(b) $V - E + F = 2$
(c) If N was even, the first player won; while if N was odd, the second player did.

(1) Explain why (a) is true. [Hint: every edge is next to a face, so why not go through all the faces counting all the edges of each? Or would this overcount the edges? by how much?]

(2) How about (b)? [Hint: can you think of your game board as a squashed polytope?]

(3) Say you draw $N = 5$ points on your game board. What are two possible values of E and F so that $3F = 3E$ and $8 - E + F = 2$? You can either use algebra or educated guessing. Once you know E, you know the total number of moves in the game! So, who wins the game?

(4) If you used algebra in (3), see if you can show $E = 3N + 3$ on a completed game board with N added points. Does this explain (c)?

Descartes's Idea

What about the angles you painstakingly measured on those pesky polytopes?

You may have noticed that the sum of the angles at a "pointy" vertex is $< 360°$, while the sum of angles at a "saddle" is $> 360°$. The "solid angle" of a vertex is $360°$ minus the sum of angles there. If we add all these solid angles up, we get

$$(360° \times V) \ - \ A.$$

(1) Calculate this for each of the polytopes. What do you see for the hole-free ($G = 0$) polytopes? and for $G > 0$?

(2) Let's try to explain this. In a polyon of n sides, what is the sum of the angles?

(3) Since each face is a polygon, summing your answer to (2) over all faces gives A. Why does this give $A = 360° \times (E - F)$?

(4) Now you can tell me what $360° \times V \ - \ A$ is in general! (Done by Descartes in 1620.)

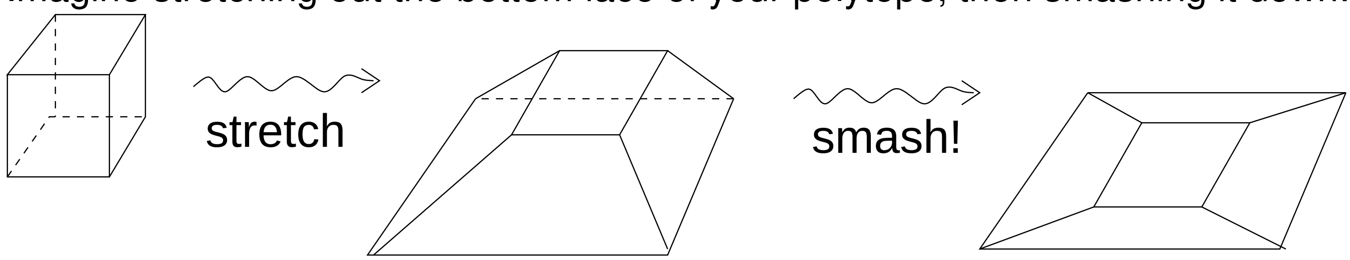
SMASHING POLYTOPES

Don't actually smash the models! OK, so if our polytope has no holes ($G = 0$), it appears that

$$V - E + F = 2. \qquad \text{(Euler's formula)}$$

This formula also came in handy explaining criss-cross winners and angle counts. But why is it true?

(1) Imagine stretching out the bottom face of your polytope, then smashing it down:



Does $V - E + F$ change if we replace the polytope by the smashed plane figure? What if we count the outside as a face?

(2) Now we'll start from an n-gon, like the stretched bottom square above, and add vertices and edges to get our smashed polytope. What is $V - E + F$ for an n-gon in the plane (counting the outside as a face)?

(3) Does $V - E + F$ change if we
(a) add an edge connecting two existing vertices? [Edges need not be straight.]

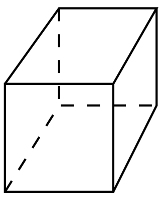
(b) add a vertex along an existing edge (splitting it into two edges)?

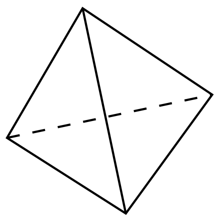
(4) Convince yourself that the operations (a) and (b) recover the smashed polytope. Putting (1), (2), and (3) together, why does this mean that $V - E + F = 2$ for the original polytope?

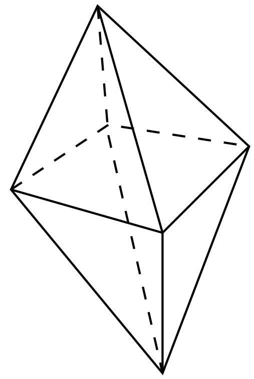
FACE PAINTING!

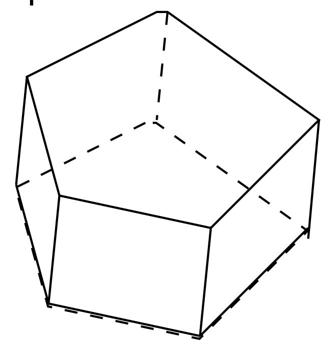
Question: How many different colors do you need to paint a hole-free polytope (each face a solid color) so that no two adjacent faces are the same color?

(1) How few colors can you use to color (in this way) each of the polytopes:









Answer: In general, a very difficult theorem says that you can always do it with *four or less*, no matter how many faces there are. With Euler's formula, we can deduce that it can always be done with six.

The way we'll do this *for a polytope with F faces*, is to assume we know (a) how to "six-color" every polytope with less than F faces, and (b) that there is always some face of your polytope with less than 6 neighboring faces. Along comes a magician and shrinks that face to a point, and (voila!) you have a polytope with one less face. By (a), you can six-color the smaller polytope. When the magician puts back in the missing face, it has five or less neighbors, so you can find a color for it that doesn't match one of the neighbors!

Let's say we believe (b). You can, of course, six-color a polytope with six or fewer faces. So with the magician's help, you can do it for any polytope with seven. But if that's true, you can do it for one with eight, and so on.

What's left is to deal with (b). We need to understand why it's impossible for every face to have six or more neighboring faces.

(2) Convince yourself that if this was true (every face has ≥ 6 neighbors), then we'd have $2E \geq 6F$, or $E \geq 3F$.

(3) Next, explain why it's always the case that $2E \geq 3V$, or $4E \geq 6V$.

(4) Euler's formula says that $V - E + F = 2$. Show that this conflicts with (2) and (3). [Hint: first, multiply both sides of the formula by 6.]