Probability Paradoxes

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February 20, 2011

1 Introduction

We're all familiar with the idea of probability, even if we haven't studied it. That is what makes probability so interesting–you don't have to know a lot of math to solve problems! Also, most people have a sense about probability, even without doing *any* math. But that sense can sometimes confuse us.

The idea of probability is simple: To each "event" A, we assign a number $P(A)$, called the *probability of* A, to represent the "chances" of that event occurring. The number $P(A)$ is always between 0 and 1. We can represent this number as either a fraction, a decimal, or a percentage. What does it mean for an event to have probability 0? What does it mean for an event to have probability 1?

Example 1.1 You find a coin on the street and you flip it. What is the probability that it will come up heads?

Solution Let H be the event that the coin is heads.

$$P(H) = \frac{1}{2} = 0.5 = 50\%$$

Problem 1.2 You roll a 6-sided die. What is the probability that it lands on 3? What is the probability that it lands on an even number?

Problem 1.3 You pick one card out of a 52-card deck. What is the probability that it is red? What is the probability that it is a 9? What is the probability that it is a red 9?

2 Conditional Probability

Sometimes, we get some new information which can change the probability. Say we know the probability of an event A, $P(A)$. Then we are given some new information–call it B. This might allow us to make a new (better) estimate for the chances of A. We call this the *probability of A given B*, and write it as $P(A|B)$.

Problem 2.1 You pick one card out of a 52-card deck. What is the probability that it is a spade? What is the probability that it is a spade given that it is black? What is the probability that it is a spade given that is a 4? What does this tell you about the relationship between suit and rank (number)?

Problem 2.2 You meet a man who says, "I have two cats." What is the probability that they are both male? Both female? What is the probability that one is male and the other is female?

Problem 2.3 You meet a woman who says, "I have two cats, and one of them is male." What is the probability that the *other* one is a male, too?

Problem 2.4 One day, you bump into me on the street while I am walking with a male cat. I tell you, "I have another cat at home." What is the probability that the cat at home is a male, too?

Problem 2.5 What if, in Problem 2.3, the woman told you that her *older* cat is a male?

3 The Monty Hall Problem

This problem is based on a game show called "Let's Make a Deal" that was on television in the 1960s and 1970s that was hosted by a man named Monty Hall. (It is still on in some parts of the world.) The game works as follows:

- There are 3 doors labeled as A, B and C. The contestant is told that behind one of the doors is a great prize (a new car). Behind the other two doors is something relatively worthless (a goat). The contestant is asked to pick the door behind which she thinks the car is hiding.

Question 3.1 Is there a "smart" choice here?

- After the contestant has made the selection, the host opens one of the other two doors and reveals a goat (i.e., he reveals that the prize is *not* behind that door).
- The host then gives the contestant the chance to switch her selection: she can either stick with her original choice, or switch to the other door (remember, there are only two remaining).
- It is revealed whether the contestant won a car or a goat.

Problem 3.2 Should the contestant switch her choice when given the chance? We will simulate the game in pairs before trying to solve this mathematically. We will make the following assumptions:

- The prize is placed behind door A, B or C in a completely random way (i.e., P(car behind Door A) $= P$(car behind Door B) $= P$(car behind Door C) $= \frac{1}{3}$).
- The host knows where the car is.
- If the contestant picks an incorrect door at the beginning, the host picks the remaining door that is incorrect. If the contestant picks the correct door at the beginning, the host eliminates one of the other doors at random.

Problem 3.3 Now suppose the host forgot to find out which door had the car behind it, so he just guessed randomly and happened to get lucky (that the door he opened wasn't in front of a car). Does this change anything?

Super Bonus Problem 3.4 You've studied the show before you went on as a contestant, and you have determined that the true probabilities are *not* as in Problem 3.2, but instead are:

$$
\begin{aligned}
P(\text{car behind Door A}) &= 0.40 \\
P(\text{car behind Door B}) &= 0.35 \\
P(\text{car behind Door C}) &= 0.25
\end{aligned}
$$

What door should you pick to begin with? What are your chances of eventually winning the car if you make this pick? Remember, you know you may well change your pick later on, so there's a lot to consider here. Solving this problem will probably take you longer than the time we have left today!

4 Expected Value

Suppose we know we perform an experiment which produces a random outcome X which can have possible values A, B or C. We want to figure out what the average value of the outcomes will be if we perform the experiment many times. We calculate the *expected value* of X by multiplying the probability of each outcome by its value. We call this $E(X)$.

$$E(X) = A \cdot P(A) + B \cdot P(B) + C \cdot P(C).$$

Example 4.1 You flip a coin. Let the "value" of heads be 1, and let the "value" of tails be 0. What is the expected value?

Solution We have $H = 1$ and $T = 0$, so the expected value is

$$\begin{aligned} &= H \cdot P(H) + T \cdot P(T) \\ &= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \\ &= \frac{1}{2} + 0 \\ &= \frac{1}{2}. \end{aligned}$$

Problem 4.2 What is the expected value if you roll a 6-sided die?

5 The St. Petersburg Paradox

Problem 5.1 We saw earlier that if you flip a coin, the probability that it will come up as heads is $\frac{1}{2}$. If I flip a coin twice, what is the probability that it will be heads both times? If I flip it 3 times? What about n times?

Problem 5.2 We are going to play a game in which the player flips a coin until a heads comes up for the first time. What is the probability that the first heads will come up on the first flip? Second flip? Third flip? What about the n^{th} flip?

Problem 5.3 In our game the player flips a coin until he gets a heads. If the first heads comes up on the k^{th} flip, then the player wins 2^k dollars. (So if you get H on the first try, you win \$2. If you get T and then H, you win \$4. If you get TTTTH, you win \$32.) What is the least a player can win in the game? What is the most? How much would *you* be willing to pay to play this game? We will simulate this game in partners and see how people do.

One way to figure out how much you should pay to play a game is to calculate the expected value of the game. We saw before that the expected value of a die roll is 3.5. So if the game is that I roll a die and I win the number of dollars that shows, I would be willing to pay any amount less than $3.50 to play, but it would not be worth it to pay more than $3.50.

Serious Problem 5.4 What is the expected value of your winnings from our coin flip game? Did you pay too much to play before? Is it *possible* to pay "too much" for this game?

6 The Envelope Paradox

A player is shown two envelopes (A and B) and told that each has money inside it and that one has twice the amount that the other one has. He is asked to pick an envelope. He picks envelope A and finds $20 inside. Then he is given the chance to switch envelopes.

We will simulate this game in pairs (with different amounts of money) and then try to understand the game mathematically.

Problem 6.1 In the $20 version of the game above, what is the expected value of the other envelope (envelope B)? Given this, should the player switch envelopes?

Problem 6.2 Is the fact that the amount in envelope A is $20 relevant to the decision? What if he had found $40? $100? $\$n$ (where n is even)?

Problem 6.3 If you answered "no" above, did the player even have to open the envelope to know he was going to switch? Does it make sense that he is "better off" picking A and switching to B instead of just picking B to begin with? What would the strategy be if he had picked B at the start?

Question 6.4 Does this problem bother you? (It bothers me very much.)

7 The Pirate Riddle

This riddle has nothing to do with probability paradoxes, but some of you may find it fun to work out (and it is one of my favorites).

- There are 7 pirates on a ship, and they have a treasure of 1000 gold coins. Not sure of how to divide it, they come up with a plan.
- They order themselves 1 through 7 randomly and decide that from pirate 1 to pirate 7–in order–each will stand before the (remaining) others and propose a plan of how to divide the coins. Then all the remaining pirates (including the one who proposed the plan) will vote on the proposal.
- Each votes "yes" or "no." If *at least* half vote "yes," then the proposal passes and they divide the coins as proposed. If *more* than half vote "no," they toss the pirate who proposed the plan overboard and the next pirate in line proposes a plan.
- Here are some assumptions you can make (and will need):
  - You can't break a coin in half (or into thirds, etc.).
  - A pirate wants to get as many coins as possible, but values his life above all else.
  - A pirate enjoys throwing another pirate overboard. So, given the choice between a situation where a pirate gets 54 coins and one where he gets 54 coins *and* gets to throw someone overboard, he will choose the latter. But he would prefer getting 55 coins to both.
  - The pirates are all perfect geniuses.

Problem 7.1 What happens? From 1 to 7 (if necessary), describe the proposal of each pirate, how the remaining pirates vote, and what subsequently takes place. This riddle is fun because the answer is somewhat surprising, yet not very difficult to come upon if you think carefully. You can email me your solutions at `jjmarshall@wustl.edu`.