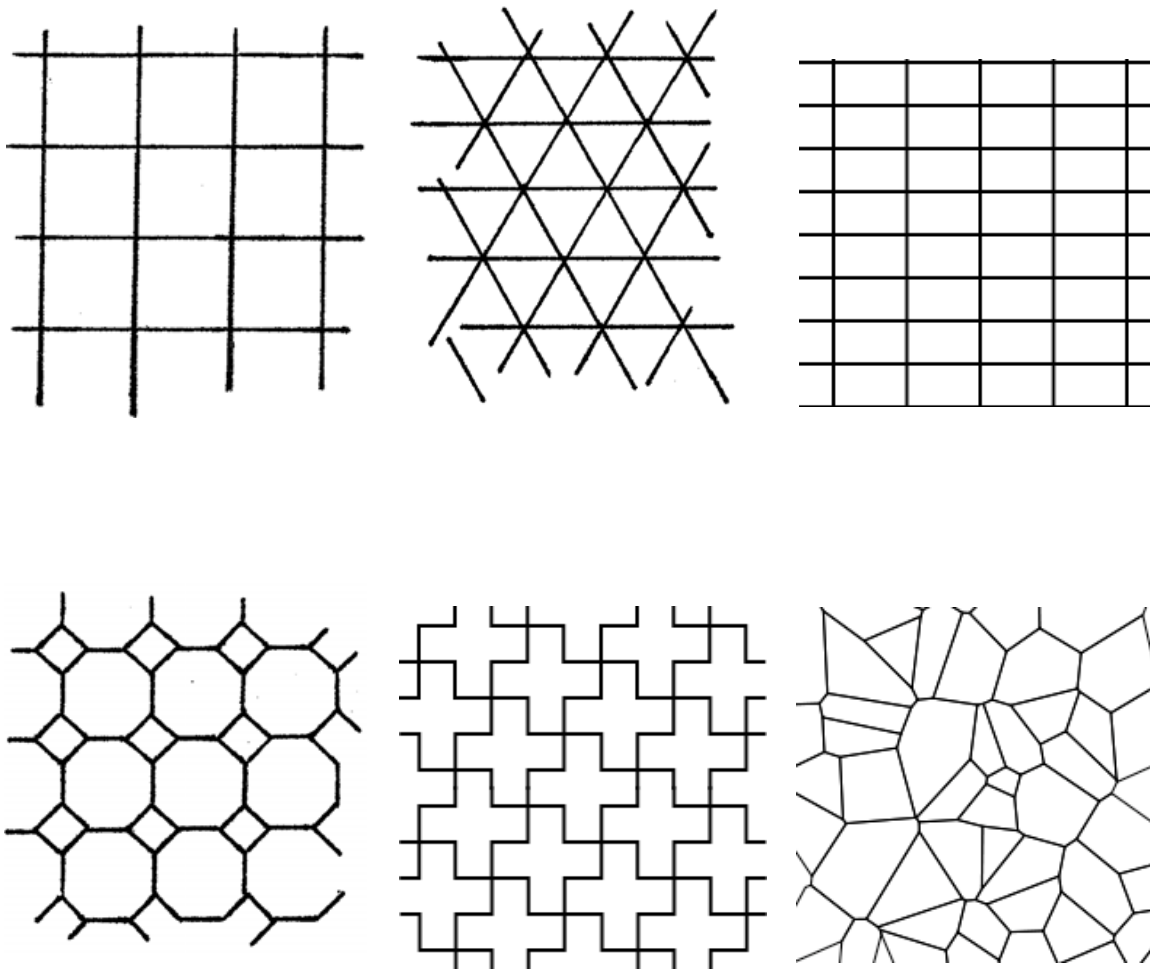


What is a tiling?

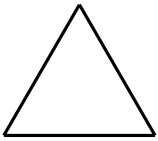
- A **tiling** refers to any pattern that covers a flat surface, like a painting on a canvas, using non-overlapping repetitions. Another word for a tiling is a **tesellation**. There are several ways to create a tiling, here are some examples.



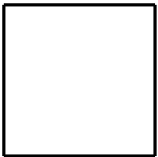
- The first two tilings are called **regular tilings** because they are made up of just one regular polygon – that is a shape having sides of the same length and equal angles – and they are placed vertex to vertex.

As mentioned above a regular tiling is made up of repeated copies of one regular polygon with the vertices of one copy only touching the vertices of another copy. A regular polygon has sides that are all the same length and equal angles.

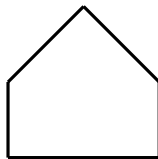
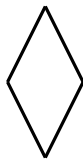
- A regular polygon with three sides is an equilateral triangle.



- A regular polygon with four sides is a square.

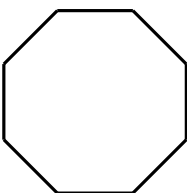


- The following shapes are not regular. Why?



Question: Can we make a regular tiling with any regular polygon?

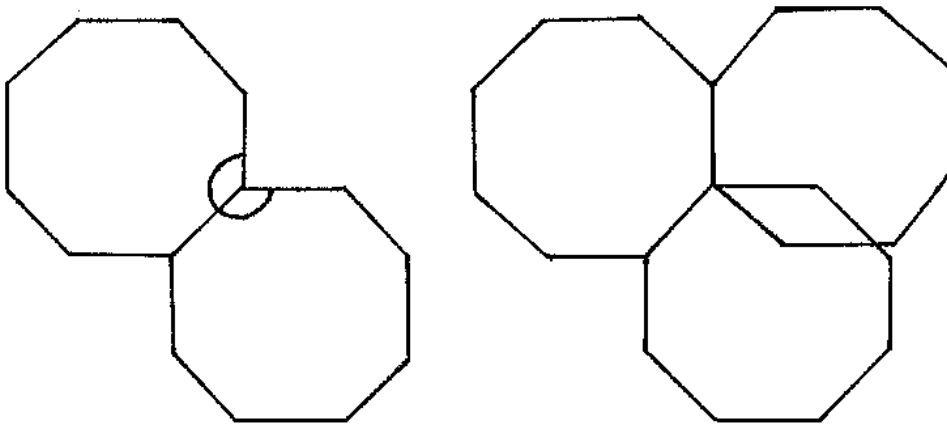
- The first two tiling on the picture in the first page show that we can make a tiling with an equilateral triangle or a square. Try to make a “stop sign tiling”, that is a regular tiling with a regular octagon.



Don't turn the page until you have tried!

Regular tilings

Don't feel bad if you were not able to succeed, because it is impossible. After placing two regular Octagons side by side there is not enough room to fit the another octagon without overlapping. Here is a picture of what is going on.

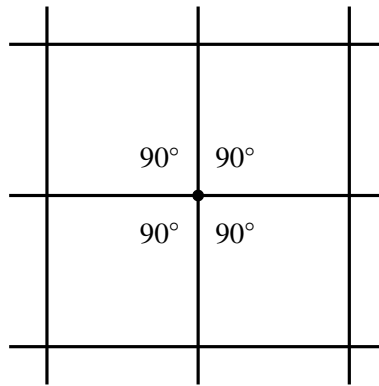


- We have seen that an equilateral triangle or a square will make regular tiling, but a regular octagon will not.

Regular tilings

Question: What regular polygons will make a regular tiling?

- To answer this question, notice that if we look at a vertex of a regular tiling the angles must add up to 360° . Since the polygons are regular, this means that the angles of our polygon must divide exactly into 360° .
- For example look at a vertex in the regular tiling formed by a square which has angles of 90° .



We have

$$\frac{360}{90} = 4$$

which is why there are 4 squares touching at the vertex.

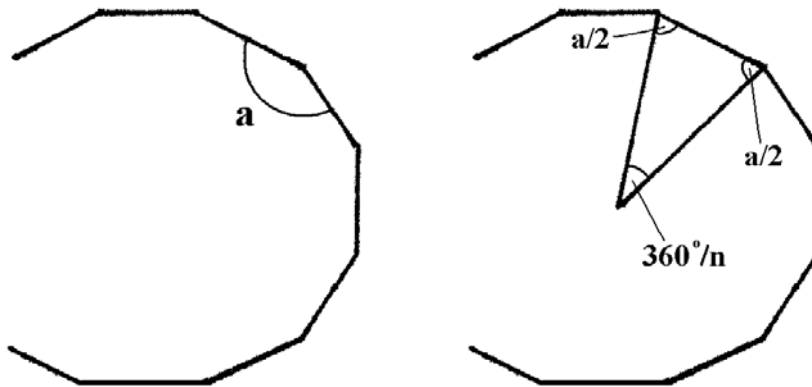
- With out looking at the picture on the first page, how many triangles will meet at a vertex of a regular tiling formed by an equilateral triangle?

Answer = _____.

- To solve our problem we must figure out which regular polygons have angles that divide 360° .

Angles of regular polygons

- What is the angle of a regular polygon? The angle should be related to the number of sides. We want to think of the polygon as having a generic number of sides n where n can be 3, 4, 5, ... (Why can't $n = 1$ or 2?). We can find a formula for the angle in terms of the number of sides, here is how.
 - Let n = number of sides of our regular polygon.
 - Let a = the angle of of our polygon.
 - Make an isosceles triangle from the center of the polygon to one of the edges.



- We know that the sum of the angles of a triangle is 180° , so

$$\frac{a}{2} + \frac{a}{2} + \frac{360^\circ}{n} = 180^\circ.$$

Solve for a:

$$a = \underline{\hspace{2cm}}$$

- Verify your formula is correct for equilateral triangles and squares which have angles of 60° and 90° respectively.

Angles of regular polygons

- Use your formula obtained on the previous page to fill out the following chart.

Number of sides, n	Angle, a
3	60°
4	90°
5	
6	
7	
8	

- Now remember that our regular polygon will make a regular tiling only when the angle, a , divides exactly into 360° . In other words, using our formula

$$a = 180^\circ - \frac{360^\circ}{n}$$

should satisfy

$$\frac{360^\circ}{180^\circ - \frac{360^\circ}{n}} = 1, 2, 3, \dots$$

Simplifying things we get

$$\frac{360^\circ}{180^\circ - \frac{360^\circ}{n}} = \frac{1}{\frac{1}{2} - \frac{1}{n}} = \frac{2n}{n-2} = 1, 2, 3 \dots$$

Regular tilings

- In other words we have turn our geometric problem into an algebra problem: Find the numbers n such that $2n/(n - 2)$ is a whole number. Fill out the following chart,

Number of sides, n	$\frac{2n}{n-2}$
3	6
4	4
5	
6	
7	
8	

- Which of the regular polygons with number of sides $n = 3, 4, 5, 6, 7,$ or 8 make a regular tiling?
- Are there any more? Substituting $n = 8, 9, \dots$ will not work because we will go on for ever. However, the answer is no. We can write

$$\frac{2n}{n-2} = 2 + \frac{4}{n-2}$$

and notice that if n is bigger than 8 then

$$\frac{4}{n-2} < \frac{4}{8-2} = \frac{2}{3} < 1.$$

This means that

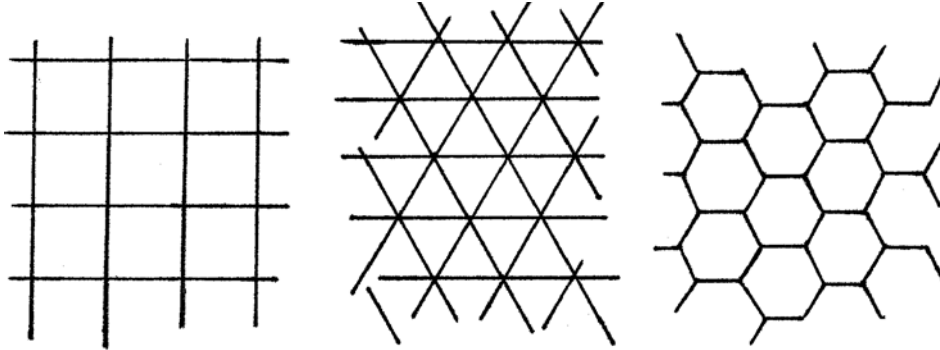
$$2 < \frac{2n}{n-2} = 2 + \frac{4}{n-2} < 3$$

when $n > 8$.

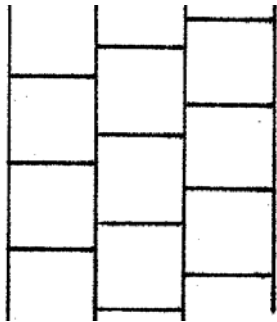
- There are no whole numbers between 2 and 3, so no regular polygon with more than eight sides can form a regular tiling.

Regular tilings

- Alas, we have solved our problem! There are only three regular polygons that form a regular tiling: equilateral triangles, squares, and regular hexagons. Here are there pictures.



- It has been said that mathematicians do not know when to “leave well enough alone”, meaning that we always find yet another wrinkle to explore. Here is a new wrinkle. Look at the tiling below:

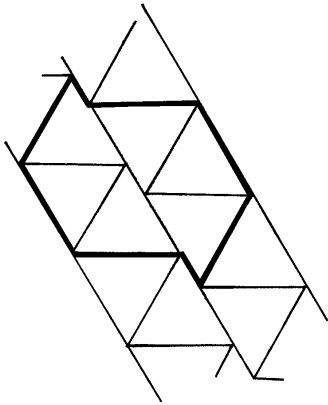


This tiling is made by sliding the regular tiling of squares.

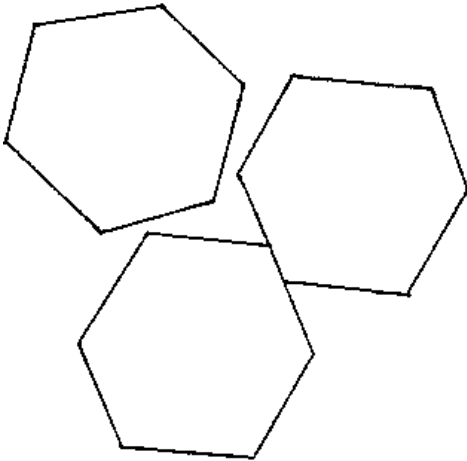
- **Questions:** Can we build a sliding tiling using equilateral triangles? What about regular Hexagons?

Sliding regular tilings

- The answers to the questions on the previous page are yes for triangles but no for hexagons. Here are a couple pictures of what is going on.
- For the equilateral triangle tiling we have parallel lines within the tiling, which we can use to slide the triangles



- However for the regular hexagons, the angles do not cooperate, leaving annoying empty spots



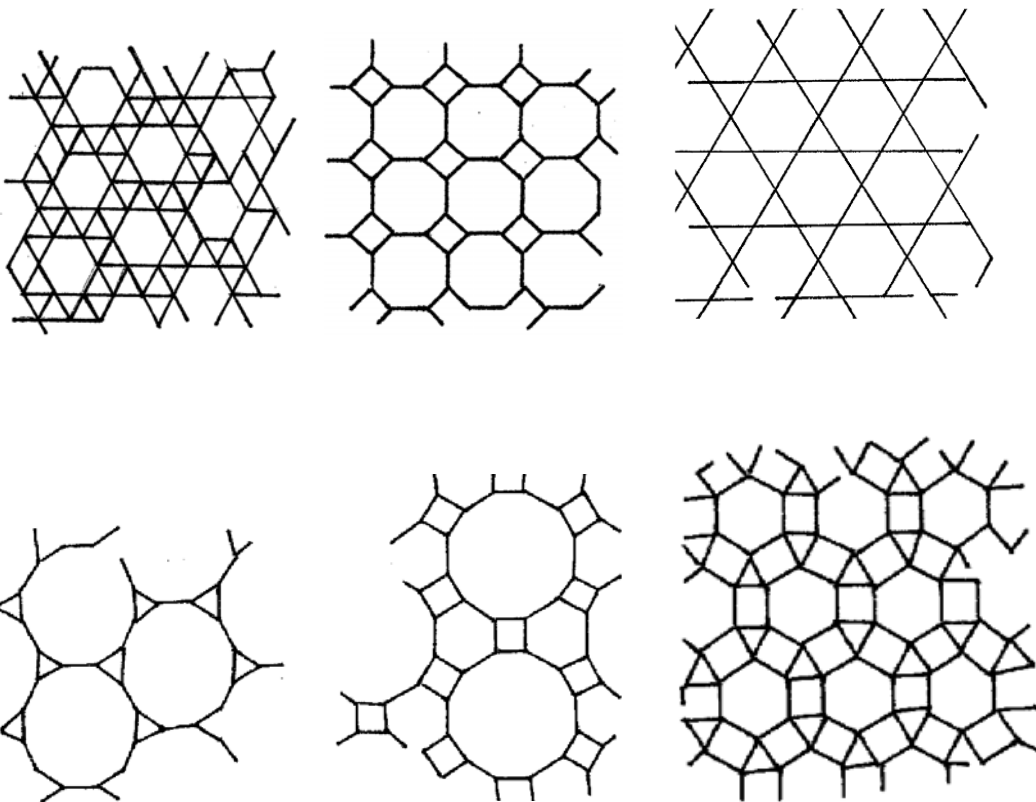
- It turns out, in order to make a sliding tiling with a regular polygon the angle of the polygon must divide 180° . So the number of sides must satisfy

$$\frac{180^\circ}{180^\circ - \frac{360^\circ}{n}} = \frac{n}{n-2} = 1, 2, 3, \dots$$

and we you can verify that this happens only for $n = 3$ and 4 .

Mixed regular tilings: more than one shape

- Okay so we have seen that there are only 3 shapes that make a regular tiling, and only two shapes that make a sliding regular tiling. What happens if we use more than one shape? For now let's keep the rule that we want the vertices to match up and that all shapes keep the same sidelength. We will call such a tiling a **mixed regular tiling**. Here are a few examples of mixed regular tilings:



- The first four use two different regular polygons and the last two use three different regular polygons.
- This begs (at least) a couple questions.

Questions: What shapes make up a mixed regular tiling? Is there a maximum number of different regular polygons we can use?

Mixed regular tilings

- It turns out the key to solving these problems is to figure out how many regular polygons can meet at each vertex. For each of the mixed tilings on previous page find how many polygons meet at each vertex. For instance, in the first picture each vertex meets four triangles and one hexagon, so 5 polygons meet. Fill in the rest of the table below.

Mixed tiling	Number of polygons at each vertex
1	5
2	
3	
4	
5	
6	

- In order to figure this out we use the same principle: at each vertex the sum of the angles meeting has to add up to 360° . Suppose that we have k regular polygons with number of sides n_1, \dots, n_k meeting at a vertex. Our basic principle means that

$$180^\circ - \frac{360^\circ}{n_1} + 180^\circ - \frac{360^\circ}{n_2} + \dots + 180^\circ - \frac{360^\circ}{n_k} = 360^\circ$$

or doing a little rearranging

$$\frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k} = \frac{k-2}{2}.$$

Now all we need to do is solve this to figure out how many shapes meet at each vertex. This is no easy task! There are too many unknowns.

- We can, however, get bounds on k . For instance we cannot have $k = 1$ or 2 (Why?). So $k \geq 3$ for a lower bound. To get an upper bound notice we must have n_1, n_2, \dots, n_k all bigger than 3 because each regular polygon has at least three sides. This makes

$$\frac{k-2}{2} = \frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k} \leq \frac{1}{3} + \frac{1}{3} + \dots + \frac{1}{3} = \frac{k}{3}.$$

- Cross multiply to get an upper bound on k , the number of regular polygons that meet at a vertex.

$$3 \leq k \leq \underline{\hspace{2cm}}.$$

Mixed regular tilings

- We have made things more manageable for our selves by showing that in order to have a mixed regular tiling, we need at least three and at most six regular polygons to meet at each vertex. However we still have to find all solutions to the equations

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} \quad k = 3$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 1 \quad k = 4$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} = \frac{3}{2} \quad k = 5$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} + \frac{1}{n_6} = 2 \quad k = 6$$

This would take a long time! Fortunately, we have computers.

- It turns out there are 17 solutions to the above equations. Here they are listed in a chart. The first equation has 10 solutions, the second has 4 solutions, the third equation has 2 solutions, and the last equation has only one solution.

	n_1	n_2	n_3		n_1	n_2	n_3	n_4	n_5	n_6
1	3	7	42	10	6	6	6			
2	3	8	24	11	3	3	4	12		
3	3	9	18	12	3	3	6	6		
4	3	10	15	13	3	4	4	6		
5	3	12	12	14	4	4	4	4		
6	4	5	20	15	3	3	3	4	4	
7	4	6	12	16	3	3	3	3	6	
8	4	8	8	17	3	3	3	3	3	3
9	5	5	10							

Chart of possibilities

	n_1	n_2	n_3		n_1	n_2	n_3	n_4	n_5	n_6
1	3	7	42	10	6	6	6			
2	3	8	24	11	3	3	4	12		
3	3	9	18	12	3	3	6	6		
4	3	10	15	13	3	4	4	6		
5	3	12	12	14	4	4	4	4		
6	4	5	20	15	3	3	3	4	4	
7	4	6	12	16	3	3	3	3	6	
8	4	8	8	17	3	3	3	3	3	3
9	5	5	10							

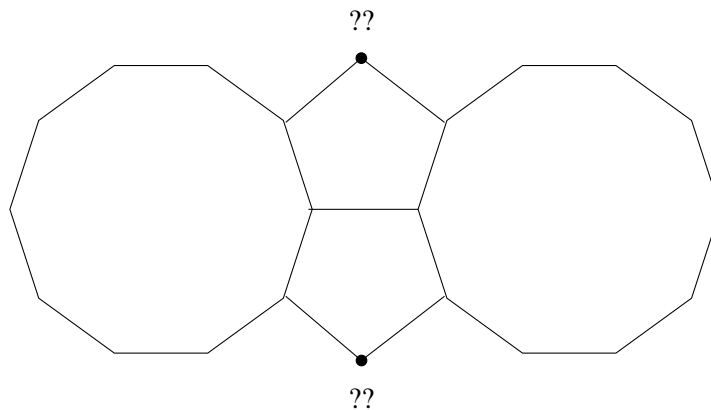
- How do you read this chart? For instance, solution **1** says that $k = 3$, $n_1 = 3$, $n_2 = 7$, and $n_3 = 42$, which verifies the identity

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{3-2}{2}.$$

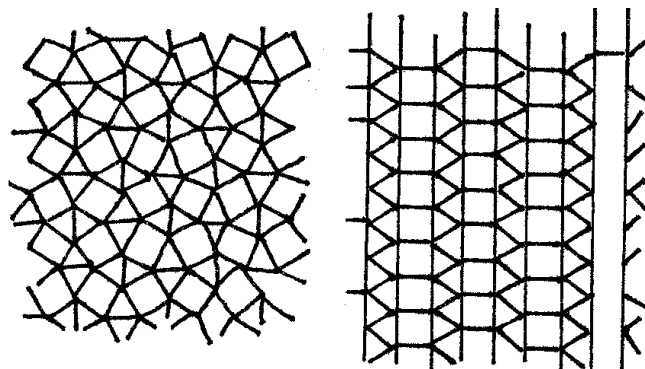
- Furthermore, solution **1** says that if we want to try to build a mixed regular tiling, at each vertex we should use one equilateral triangle ($n_1 = 3$), one heptagon ($n_2 = 7$), and one regular polygon with 42 sides ($n_3 = 42$). Solution **8** says that we should have one square pentagons ($n_1 = 4$) and two octagons ($n_2 = n_3 = 8$) meeting at each vertex.
- Which solutions in the chart correspond to regular tilings, that is only using one regular polygon? (Hint: there are three of them).
- What is the maximum number of different shapes one can use for a regular mixed tiling?

Mixed regular tilings

- It turns out that not every solution above will produce a mixed regular tiling. The problem is that these combinations work at one vertex, but they cannot be repeated again and again at each vertex, forming a tiling.
- For instance, solution **9** uses two regular pentagons and one regular decagon. Below, you can see what happens if you try use this configuration. Try to complete the picture, remember at each vertex we must have two regular pentagons and one regular decagon. What happens at the marked vertices?



- Likewise the solutions **1**, **2**, **3**, **4**, **6** and **11** do not produce tilings either (Try to draw these!). The solutions **10**, **14**, and **17** correspond to regular tilings, so we disregard them. This leaves seven other true regular mixed tilings: **5**, **7**, **8**, **12**, **13**, **15**, and **16**.
- The catch is that the polygons may be arranged in more than one way. For example solution **15** has the following possible arrangements.



The Mathematics of Escher

